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A THEORETICAL FRAMEWORK FOR DESIGN OF ROBOTIC FIXTURES

Ph.D. Dissertation in Mechanical Engineering
by
Walter Willem Nederbragt

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Walter Willem Nederbragt

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DISSERTATION

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Approved:

Bahram Ravani

Steven A. Velinsky

Harry Cheng

Committee in Charge

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Abstract

The relative location of a robot's end- effector to objects in the robot's workspace is sometimes unknown. Tactile sensing fixtures, when used with a touch probe attached to the robot, can determine the relative location of these objects. Currently, the work on the design of tactile fixtures for this purpose has been limited. This dissertation addresses this issue by creating a theoretical framework for the design of tactile fixtures. In this framework, fixtures are analyzed based on the geometric surfaces that compose them and the contacts that are made to them.

The analysis of fixtures based on their geometric surfaces relies upon the Euclidean group and its subgroups. Using these groups, several propositions are introduced and proven. These propositions form the basis of a new theory that can aid in the design of touch sensing fixtures by analysis of the continuous and finite groups that represent them. Using these propositions, fixtures involving different geometric elements are analyzed for their usefulness in determining the relative position between two bodies.

This analysis is taken one step further by looking at contacts needed to make a "useful" reference fixture. Contacts between spheres, planes, cylinders, points, and lines are studied, and group representations are found for every possible contact

that could exist between these geometric elements. Using the group representations, all possible combinations of contacts are studied. During this enumeration, 17,465 "useful" contact combinations are found.

Finally, using the information obtained from the contact analysis, two simple yet novel touch sensing fixtures for referencing are developed. One of those fixtures uses a plane-cylinder geometry to uniquely locate a frame. The other fixture uses a tripod shaped probe and a planar surface (in the final design a digitizer is used) to uniquely locate a reference frame.

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To my parents,

who made it all possible.

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Chapter 1

Introduction

In the last fifty years, our view and use of robots has evolved from the stuff of science fiction films to the reality of computer controlled actuators integrated into a wide variety of devices for use in many different environments. Regardless of the environment, robots always need to know two things to be useful, they need to know the location of their end-effector relative to their base, the determination of this location is commonly referred to as calibration, and the location of any parts in the workspace that will be manipulated, the determination of the part locations is commonly referred to as workspace referencing or part referencing.

Part referencing is the process of determining the relative location of a part with respect to a tool (such as a machine tool, a robot, or a material handling system) or with respect to a world coordinate system (see Figure 1.1). Part location data is necessary for automated machine tool programming and part processing. In manu-

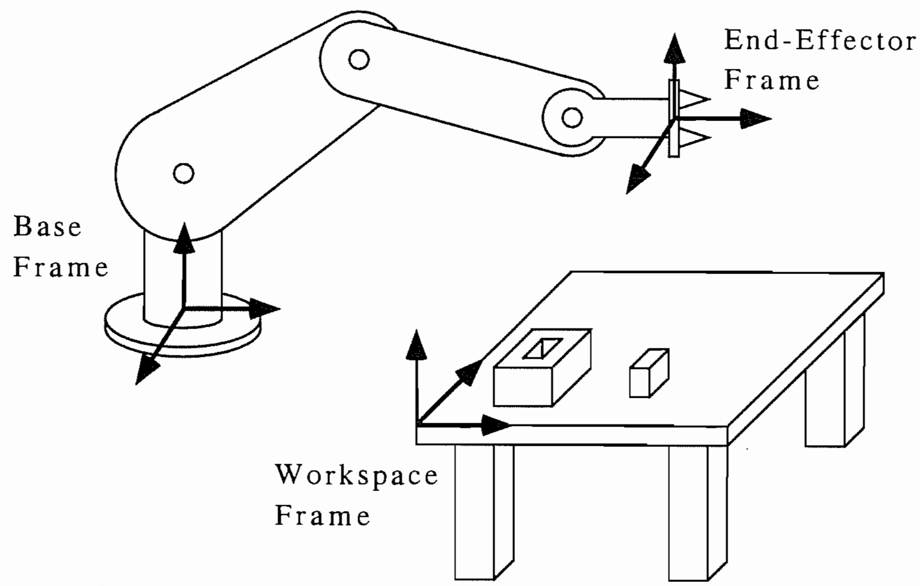


Figure 1.1: Relationship between frames

facturing, mechanical fixtures have been designed (see, for example, [10] or [38]) that would allow repeatable positioning of a pallet with respect to a machine tool at a pre-determined location.

In robot calibration, the position of the end-effector is usually measured at a set of pre-determined locations using some form of a sensing system (see Figure 1.1). This data is then combined with joint encoder readings from the same set of locations to update the kinematic parameters of the robot in its programming system (see, for example, [36] or [16]) to improve its positioning accuracy.

Since both part referencing and calibration require measurement of relative locations between two objects, mechanical fixtures are usually used to simplify the sensing function and to improve repeatability. There are also approaches that have relied on directly measuring elements of feature surfaces of the parts eliminating the need for mechanical fixtures. These approaches have usually been based on the use of non-contact type sensing systems such as use of theodolites in robot calibration [43] or laser interferometry (see for example, [13] or [19]). Mechanical fixtures, however, are used (most of the time) in conjunction with touch or tactile sensing.

1.1 Motivation

Much of the existing work related to tactile sensing fixtures have reported one of a kind and ad hoc systems. There has been very little effort on developing a broader design method or theory for such fixtures or for better understanding of key design parameters. An exception to this is the work of McCallion and Pham [23] in relationship to their work in developing sensing fixtures for robotic assembly.

McCallion and Pham used kinematic mobility criterion to systematically determine the number of touches necessary for different sensing arrangements using faces of a cubical element to determine the location of an end-effector with respect to a cubical fixture. Such cube shaped tactile sensing fixtures have also been used in robot calibration by Mooring and Pack [26]. Other common shapes used for the mechanical fixtures are three spheres [10] [38].

Although a cubical fixture and a three sphere fixture can be used for referencing, they are only two of many different reference fixture geometries. Moreover, these two fixtures are designed based on the use of a touch probe that makes point contacts on the surfaces of the fixture to determine the location of the fixture; a point-surface contact is only one of an infinite number of contact possibilities.

1.2 Overview of the Dissertation

In this dissertation a relatively general theoretical foundation is developed that aids in the design of tactile sensing fixtures using group theory by exploiting the symmetry of different measuring arrangements. This group theoretical foundation includes the analysis of fixture geometries that form reference fixtures and the analysis of the contacts made to the fixture geometries for determination of the location of the fixture. The use of group theory is appropriate since the idea behind part referencing is to determine the relative displacement between two parts which forms the well known Euclidean group or one of its sub-groups. The organization of the paper is as follows.

Chapter two describes related work that has an impact on the work described in this dissertation. Including important work by Hervé [15], Torras and Thomas [41], and Liu and Popplestone [21].

Chapter three discuss a few relevant aspects of symmetry groups, describes a systematic method for finding the continuous subgroups of the Euclidean group, and introduces and proves several propositions that form the basis of our new theory that can aid the design of touch sensing fixture systems. Then, mechanical fixtures with surfaces consisting of spheres, planes (this will include cubes with planar faces), right cylinders and their combinations are studied, using group theory and the propositions developed, for use as touch sensing fixtures.

Chapter four uses group theory for type synthesis or enumeration of contacts between geometric elements necessary in the design of tactile sensing mechanical fixtures for robotic applications. Although the scope of this dissertation is limited to geometric contacts involving points, planes, spheres, lines, and right cylinders, the techniques developed are general and can be applied to other geometric features and non tactile sensing elements used in robotic referencing and calibration.

Chapter five discusses two simple yet novel touch sensing fixture for part referencing and calibration in manufacturing and robotics that originate from our study of fixture design.

Chapter six evaluates the significance of this work, its contributions and limitations, and envisions possible future research based on what has been accomplished.

Chapter 2

Existing Literature

Fixture design for referencing and calibration is related both directly and indirectly to many research fields. For example, analysis of feature interactions are discussed in the field of robotic assembly, and motion analysis is frequently discussed in the field of mechanisms. In the proceeding sections an introduction is given to existing literature on the subjects pertinent to this dissertation.

2.1 Mechanism Analysis Using Group Theory

Hervé [15] introduced a classification of mechanisms by applying the theory of continuous groups. He used the subgroups of the Euclidean group to represent each joint in a mechanism. Using the dimension of the subgroups that corresponded to each mechanical joint, he formulated equations making it possible to analyze a series of mechanism connections (composition relation) and a set of parallel mechanism connections (conjoined relation).

Hervé represented the intersection and composition of constraints in terms of groups. If there are two relations L_1, L_2 between $body_1$ and $body_2$ then the conjoined relation of $body_1$ and $body_2$ is $L_1 \cap L_2$. When relations are composed, one has the following relationship between the dimensions:

$$dim\{L(i,k)\} = dim\{L(i,j)\} + dim\{L(j,k)\} - dim\{L(i,j) \cap L(j,k)\} \tag{2.1}$$

where i,j,k refer to three distinct bodies. With his analytical tools, Hervé made it possible to determine the degree of freedom of any mechanism. Furthermore, he created a classification of mechanisms based on his results.

Fanghella, Galletti and others (see, for example, [11], [12], and [2]) have extended Hervé's research using a combination of group theory and geometry in the analysis of mechanisms and robot manipulators.

This work is relevant to the analysis of touch sensing fixtures because the interactions of a touch sensor and a fixture are similar to joints in a mechanism. In fact, the contact between different geometric elements can be represented using mechanical joints. This relationship is explored further in Chapter four.

2.2 Robotic Assembly Planning

In 1980, Popplestone applied the theory of continuous groups to robotic assembly planning to obtain new results [34]. He observed that features of a body are useful

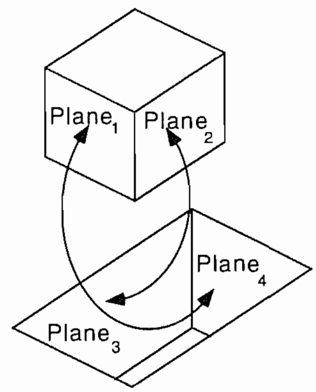


Figure 2.1: Constraints between bodies during assembly

descriptors, perhaps even more important than the symmetry group of the body itself. In his analysis, he treated a body as a set of infinite primitive surfaces where the surfaces bounded the body or solid. This made interactions between bodies simpler because just the surfaces of the bodies in contact need to be studied to determine the possible motions between the bodies. With his results, he modified his robot programming language RAPT to use his methods for solid body interactions in robot task planning.

Thomas and Torras [41] extend Popplestone's ideas to find the configuration or configurations that satisfy all constraints between a given set of bodies (see Figure 2.1). Their method is based on the symbolic manipulation of chains of matrix products. The matrices used are derived from the theory of continuous groups applied to the Euclidean group, commonly denoted as $SE(3)$. They tabulated the outcomes of intersections and multiplications of certain continuous groups of $SE(3)$.

Liu and Popplestone [20] [21] have continued research in this area by studying a family of the subgroups of the proper Euclidean group. Each member of this family is called a TR group since it is a semi-direct product of a translation group T and a rotation group R. Some examples of TR groups are the translation subgroups T^1, T^2, and T^3, the rotation subgroups $SO(2)$, $SO(3)$, and $O(2)$, platonic groups, planar motion groups, and cylindrical motion groups.

They have used their analytical results to develop a program $KA3$. This program is an assembly planning system. $KA3$ uses geometric boundary models of assembly components provided by a geometric solid modeller as input. It then finds a set of detailed robotic assembly task specifications using its programmed knowledge of geometry and symmetry.

An understanding of the work on robotic assembly planning is essential for the analysis of tactile fixtures because the assembly of components is similar to the necessary "constraint" of a reference fixture in Euclidean space for determination of the relative location between the fixture frame and robot. This relationship is explored further in Chapter three.

2.3 Robotic Calibration and Part Referencing

Although referencing fixtures have been used for many years, there is a limited amount of published papers on the design of these fixtures. In 1984, McCallion and Pham developed a procedure for finding the relative location of a robot manipulator to a cube shaped fixture using tactile and force/moment sensors [23]. Duffie et al. [10]

developed a procedure similar to McCallion's and Pham's, however, three spheres were used in place of the cube as the reference fixture and only tactile sensors were used. Mooring et al. [27] described a calibration fixture that determined the relative position of the robot manipulator to a fixture by aligning a cube attached to the manipulator with three perpendicular surfaces on the fixture. Each of these methods are described in more detail in the following sections.

2.3.1 Cubical Referencing Fixtures

McCallion's and Pham's [23] procedure for find the relative location of a robot manipulator's frame to the frame of a body in three dimensional Euclidean space uses either tactile or force/moment sensors. In both cases, the robot's manipulator is equipped with a touch sensitive wand, and the body is outfitted with a cube shaped fixture. The cube shaped fixture can be supplied with either a force/moment sensor in its base or touch sensors on its surfaces.

When the cube shaped fixture is equipped with a force/moment sensor in its base, the sensor can feed back the information necessary to calculate the force "line of action." Using the fixture with the force/moment sensor, the location of a touch on the fixture can be calculated. McCallion and Pham found that three touches to the fixture is sufficient for the calculation of the relative location of the two frames.

McCallion and Pham also found that the relative location of the cube to the manipulator can be found by touching the touch sensitive surfaces of the cube with

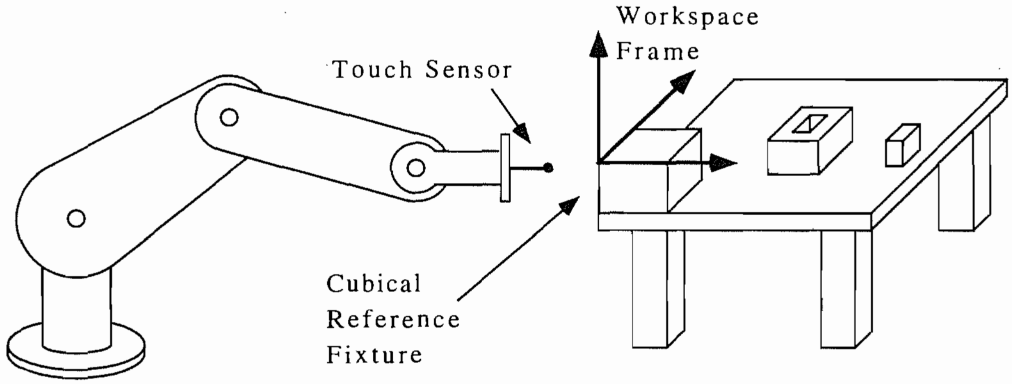


Figure 2.2: Cubical reference fixture

the manipulator's wand in six different locations, two touches on three sides of the cube (Figure 2.2). Using the location of these touch points, in the manipulator's frame, the location of the cube's sides can be calculated. By finding the relationship between the three edges of the cube and the frame of the manipulator, the relative location can be determined.

2.3.2 Three Sphere Referencing Fixtures

Duffie et al. [10] developed a procedure similar to McCallion's and Pham's, however, three spheres were used in place of the cube as the reference fixture and the fixture was only equipped with touch sensitive surfaces. In this case, the three spheres were touched with the robot manipulator's touch sensitive wand. They found that if each sphere was touched four times then the location of the center of the sphere could be found, and if this was done on each of the three spheres, the relative location of the three sphere fixture to the robot manipulator could be found (Figure 2.3).

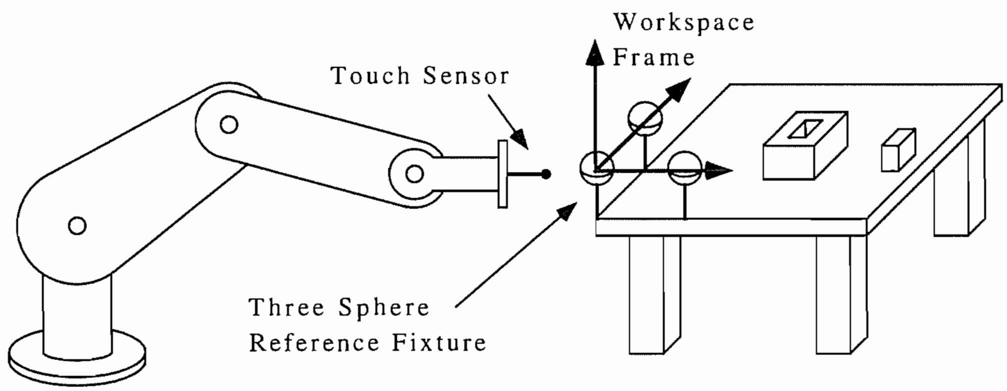


Figure 2.3: Three sphere reference fixture

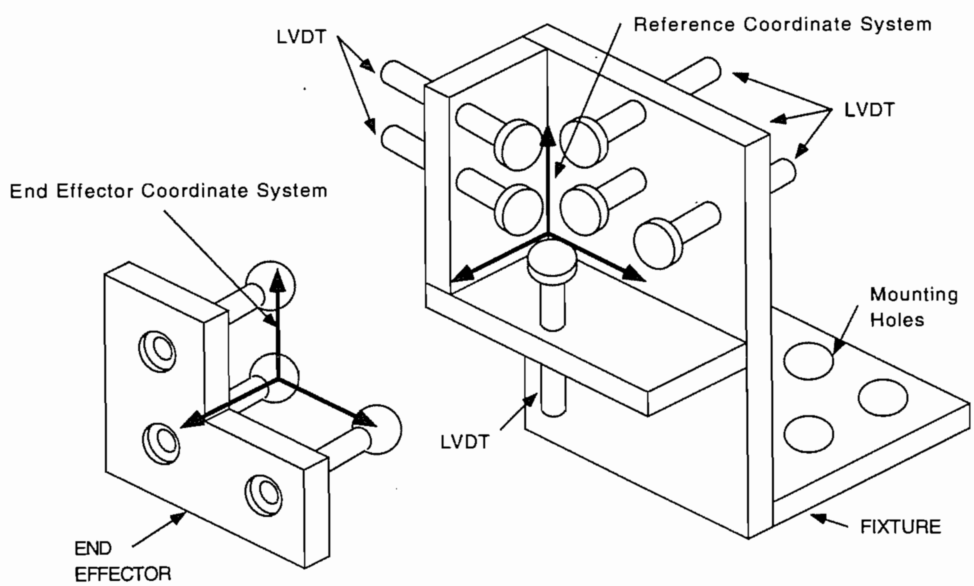


Figure 2.4: Full pose reference fixture

2.3.3 Existing Calibration Fixtures

Mooring et al. [27] described a completely different fixture that determined the relative position of the manipulator to a fixture by aligning a solid body attached to the robot manipulator with three perpendicular surfaces on the fixture. The solid body, having a known shape, uniquely describes the location and orientation of the end effector of the robot when the solid body touches all three perpendicular surfaces of the fixture. Figure 2.4 illustrates this idea using six LVDT's (the six LVDT's are equivalent to the three perpendicular surfaces) to locate the solid body attached the end effector of the robot.

Chapter 3

Group Theory and Referencing

In this chapter a few relevant aspects of group theory and symmetry groups are discussed and several propositions are introduced and proven. These propositions form the basis of a new theory that can aid the design of touch sensing fixture systems. Then, mechanical fixtures with surfaces consisting of spheres, planes (this will include cubes with planar faces), right cylinders and their combinations are studied using group theory and the propositions developed. The result being the development of new and useful touch sensing fixtures.

3.1 Group Theory

Group theory was originally developed in the nineteenth century to measure symmetry. Therefore, it is commonly used when looking at the symmetry of an object [44]. In the proceeding paragraphs, the definitions for a group, a subgroup, and order of a group are given. To aid in the understanding of group theory, the set of real numbers and the set of integers are proven to be a group and subgroup, respectively, using addition as the binary operation. For more information on group theory the interested reader is referred to [3], [4], [8], and [22].

Definition 3.1 *A* **group** *is a set G together with an operation defined between pairs of elements $a, b \in G$ (the operation is a binary operation, and it is commonly referred to as multiplication – the operation is usually written as ab.) which satisfy the following axioms:*

1. *$\forall a, b \in G, ab \in G$ (closure).*

2. *$\forall a, b, c \in G, (ab)c = a(bc)$ (associative).*

3. *There exists an element i where $\forall a \in G, a(i) = (i)a = 1$ (identity).*

4. *For each $a \in G \; \exists \, a^{-1}$ such that $a(a^{-1}) = (a^{-1})a = i$ (inverse).*

Definition 3.2 *A subset of a group is a* **subgroup** *if all of the properties of a group are satisfied (i.e., closure, identity, and inverse).*

Definition 3.3 *The* **order** *of any group G is the number of its elements. This is often denoted as $|G|$ = number of elements.*

Proposition 3.1 *If we let $G = \{x | x \in \mathbb{R}\}$, and we let addition be the binary operation (called multiplication), then, G is a group, and the set $S = \{x | x \in I\}$ is a subgroup of G.*

proof: *Let $a, b \in G$. The element ab is always an element of the real numbers, the real numbers are associative, zero is the identity element, the inverse of an element is the negative of that element (which is also a real number), therefore, G is a group. S is a subgroup because two integers add to become another integer, the inverse of an integer is the negative of the integer, and the identity element, zero, is in S.*

Proposition 3.1 proves that real numbers and integers, using addition as the binary operation, form a group and a subgroup respectively. However, the real power of group theory comes when it is applied to objects to evaluate their symmetry. In the next section symmetry groups, the proper Euclidean group, and displacements (which are elements of the Euclidean group) are described and applied.

3.2 Euclidean Group and Symmetry Groups

Symmetry groups, also known as permutation groups, are used to measure an objects symmetry. In the past they have been used to study molecular structures in chemistry [44]. More recently, the Euclidean group, a symmetry group, has been used in mechanical design to study geometric relationships between solids. In this section the definition of a symmetry group is given and the Euclidean group and its elements are described.

3.2.1 Theory

Definition 3.4 *let $S(X)$ be the set of bijections from X into itself, then $S(X)$ together with the operation of composition form a group called the* **symmetry group** *or the permutation group of S.*

Definition 3.4 does not describe the binary operation used to make a symmetry group because the binary operation used is independent of the definition. If an operation can be found that satisfies the definition of a group then it can be used. One such binary operation is the displacement operation. This operation is commonly used in the fields of robotics and mechanism design to represent changes in location of solids [9]. A displacement in these fields is usually represented using a matrix transformation. Equation 3.1 is the matrix representation for a displacement. Equation 3.2 is the homogeneous matrix representation for a displacement [28].

$$\begin{bmatrix} x_1 \\ y_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}. \tag{3.1}$$

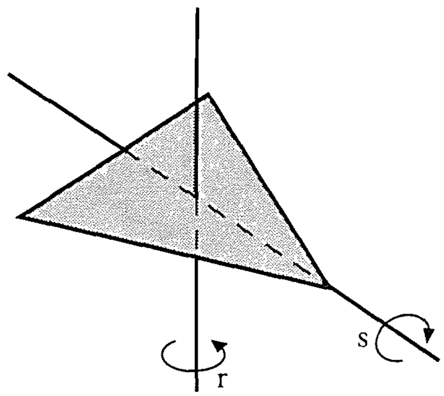
$$\begin{bmatrix} x_1 \\ y_2 \\ z_3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{31} & a_{32} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 1 \end{bmatrix}. \tag{3.2}$$

If the displacement operation is applied to the entire Euclidean space of dimension three, it forms a special symmetry group called the proper Euclidean group $SE(3)$. This group contains all possible displacements of a body in the Euclidean space. Therefore, all possible displacements that map a solid back to itself form a subset of $SE(3)$ and, from the definition of symmetry, must form a subgroup of the Euclidean group. Hence, all solids have a symmetry group, using the displacement operation, that represents their symmetry. In the case where an object has no symmetry at all, the object's symmetry group is only the identity element (commonly denoted as $\{I\}$). The binary operation used in this dissertation for analysis of objects will always be the displacement operator.

3.2.2 Examples

Objects can have a symmetry group of finite order or infinite order. If an object has a symmetry group of finite order then the object can only be rotated into a finite number of positions. An example of this is an equilateral triangle, the triangle can be rotated about its center and it can be flipped. In Figure 3.1, an equilateral triangle is shown with two axes of rotation. A group multiplication table is also given that shows all possible orientations for the triangle. As can be seen form Figure 3.1, the triangle has a finite order of six because their are six elements in the symmetry group.

Many objects have infinite order symmetry groups. For example, a sphere can be rotated about any axis through its center, and it will be mapped to itself (Figure 3.2).



G = {e, r, rr, s, rs, rrs}

Multiplication Table

	e	r	rr	s	rs	rrs
e	e	r	rr	s	rs	rrs
r	r	rr	e	rs	rrs	s
rr	rr	e	r	rrs	s	rs
s	s	rrs	rs	e	rr	r
rs	rs	s	rrs	r	e	rr
rrs	rrs	rs	s	rr	r	e

Figure 3.1: An equilateral triangle in $SE(3)$, its group, and the group multiplication table.

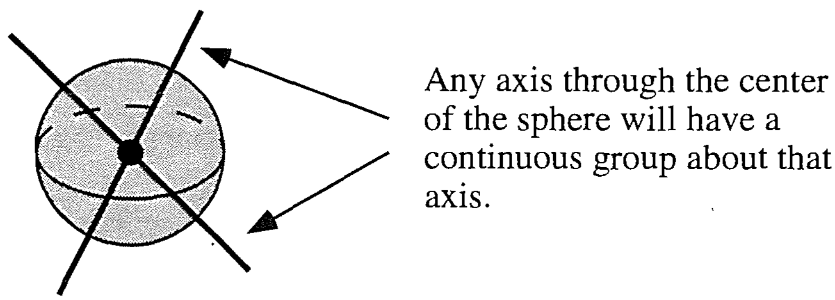


Figure 3.2: A sphere has infinite symmetry about any axis through its center.

On the other hand, a cylinder has only one axis of continuous or infinite symmetry. This axis of rotation is coincident with the center line of the cylinder. The cylinder, unlike the sphere, has an infinite number of axes with finite symmetry (Figure 3.3). These axes are perpendicular to the center line of the cylinder and they go through the mid-point of the cylinder's center line.

As stated earlier all solids have a symmetry group and that symmetry group is a subgroup of the Euclidean group $SE(3)$. $SE(3)$, which contains all possible translations and rotations in Euclidean space, has an infinite number of subgroups. The

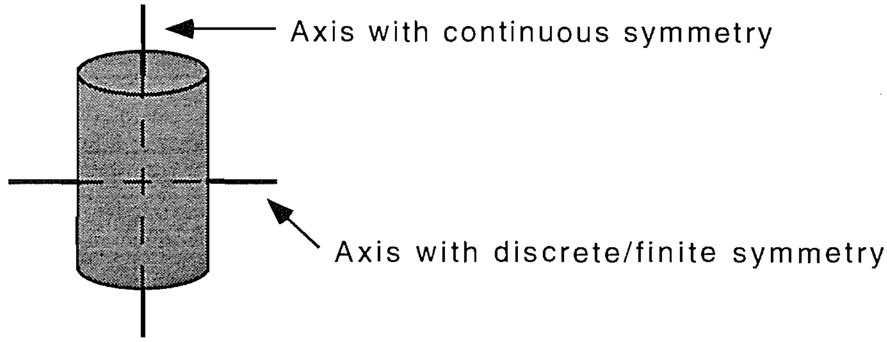


Figure 3.3: A cylinder has infinite symmetry about one of its axes and finite symmetry about all of the others.

continuous subgroups of $SE(3)$ can be described by their conjugation classes. There are twelve such classes. Each class represents a type of continuous subgroup in $SE(3)$. Some examples of these subgroups are: planar displacements, pure translations, pure rotations, and one dimensional translations. The most significant of these subgroup classes are the six mechanical lower pair joints(Figure 3.4), the lower pair joints are the joints with surface to surface contacts.

These joints are common in many mechanical designs. For example, Hervé [15] was one of the first to take advantage of the symmetry group $SE(3)$ and its subgroups to analyze mechanisms using these subgroups. He found a way of determining the number of degrees of freedom of a mechanism. In the next section all of the classes of continuous subgroups of $SE(3)$ will be found using Lie Algebras and Lie Groups. These subgroups will later be used for analysis of fixtures.

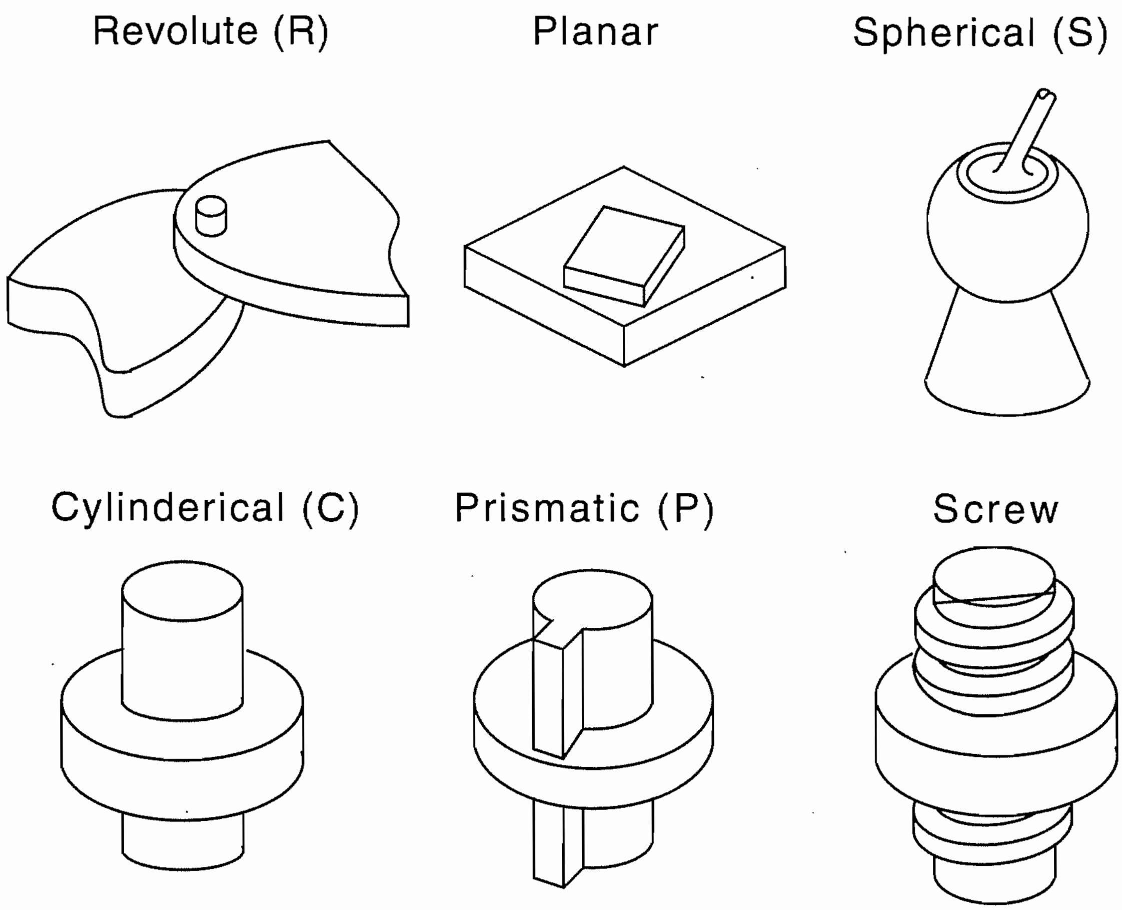


Figure 3.4: The six mechanical lower pair joints.

3.3 Finding the Continuous Subgroups of $SE(3)$

In this section Lie algebras and Lie groups are used to find the classes of subgroups of $SE(3)$. In order to find these classes, a brief introduction to Lie Algebras, Lie groups, the Lie algebra and Lie group of $SE(3)$, and the Lie Algebra and Lie group of $SO(3)$ are given. Then, the Lie algebra of $SE(3)$ is broken down in a systematic method in order to find all of the subgroup classes. For more information on Lie groups and Lie algebras refer to Varadarajan [42]. For more information on the Lie algebra and Lie group of $SE(3)$ and $SO(3)$ the interested reader is referred to [17] [29].

3.3.1 Definitions

Definition 3.5 *A vector space V (over $\mathbf{R}$) is a* **Lie algebra** *if there exists a bilinear operator $V \times V \to V$, denoted $[\cdot,\cdot]$, satisfying:*

1. *Skew-symmetry:* $[v, w] = -[w, v] \ \forall\, v, w \in V$.

2. *Jacobi identity:* $[[v, w], z] + [[z, v], w] + [[w, z], v] = 0$.

Definition 3.6 *A subspace $W \subset V$ is called a* **Lie subalgebra** *if $[v, w] \in W \ \ \forall v, w \in W$.*

Definition 3.7 *A subspace $H \subset V$ is called an* **ideal** *if $[v, h] \in H \, \forall v \in V$ and $\forall\, h \in H$.*

Definition 3.8 *A* **Lie group** *is a group G which is also a smooth manifold and for which the group operation $(g,h) \to gh$ and $g \to g^{-1}$ are both smooth.*

The group of rigid transformations on $\mathbb{R}^3$, $SE(3)$, is defined as the set of mappings $g : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $g(x) = Rx + p$, where $R \in SO(3)$ and $p \in \mathbb{R}^3$. An element of $SE(3)$ is written as $(R,p) \in SE(3)$. $SE(3)$ can be identified with the space of 4×4 matrices of the form

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$. $SE(3)$ is a Lie group of dimension 6.

The Lie algebra of $SO(3)$, denoted $so(3)$, may be identified with the 3×3 skew-symmetric matrices of the form:

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -w_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3.3}$$

with the bracket structure

$$[\widehat{\omega_1}, \widehat{\omega_2}] = \widehat{\omega_1}\widehat{\omega_2} - \widehat{\omega_2}\widehat{\omega_1}, \quad \widehat{\omega_1}, \widehat{\omega_2} \in so(3). \tag{3.4}$$

The Lie algebra $so(3)$ can be identified with $\mathbb{R}^3$ using the mapping in equation 3.3, which maps a vector $\omega \in \mathbb{R}^3$ to a matrix $\widehat{\omega} \in so(3)$. It can be shown that

$$[\widehat{\omega_1}, \widehat{\omega_2}] = (\omega_1 \times \omega_2)^\wedge, \quad \omega_1, \omega_2 \in \mathbb{R}^3. \tag{3.5}$$

Hence $\omega \mapsto \widehat{\omega}$ is a Lie algebra isomorphism between the Lie algebra $(\mathbb{R}^3, \times)$ and the Lie algebra $(so(3), [\cdot,\cdot])$.

The Lie algebra of $SE(3)$, denoted $se(3)$, can be identified with 4×4 matrices of the form:

$$\widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \quad \omega, v \in \mathbb{R}^3 \tag{3.6}$$

with the bracket structure

$$[\widehat{\xi}_1, \widehat{\xi}_2] = \widehat{\xi}_1 \widehat{\xi}_2 - \widehat{\xi}_2 \widehat{\xi}_1. \tag{3.7}$$

If we let

$$\widehat{\xi}_1 = \begin{bmatrix} \widehat{\omega_1} & v_1 \\ 0 & 0 \end{bmatrix} \quad and \quad \widehat{\xi}_2 = \begin{bmatrix} \widehat{\omega_2} & v_2 \\ 0 & 0 \end{bmatrix} \quad then,$$

$$[\widehat{\xi}_1, \widehat{\xi}_2] = \widehat{\xi}_1 \widehat{\xi}_2 - \widehat{\xi}_2 \widehat{\xi}_1 = \begin{bmatrix} (\omega_1 \times \omega_2)^\wedge & \omega_1 \times v_2 - \omega_2 \times v_1 \\ 0 & 0 \end{bmatrix}. \tag{3.8}$$

The vector space $se(3)$ is isomorphic to $\mathbf{R}^6$ via the mapping $\widehat{\xi} \mapsto \xi = (v, w) \in \mathbb{R}^6$.

3.3.2 Derivation of Subgroup Classes

Proposition 3.2 *Translations T form an ideal in $se(3)$.*

proof: *Let X_1 and Y_2 represent two displacements where $X_1 = (x_1; y_1) \in se(3)$ and $X_2 = (0; y_2) \in T$ (this is a screw representation for displacements [7] [24]). $[X_1, X_2] = X_1 \times X_2 = (x_1; y_1) \times (0; y_2) = (0; x_1 \times y_2) \in T$. Therefore, T is an ideal from the definition of ideal.*

IF given a Lie algebra V and an ideal H of V, then V/H is also a Lie algebra [42]. Since $se(3)$ is a Lie algebra and T is an ideal, then $se(3)/T$ must also be a Lie algebra. From [17] it is known that $se(3)/T = so(3)$. Figure 3.5 shows the mapping $\pi : se(3) \mapsto se(3)/T = so(3)$. The Lie algebra $so(3)$ is of dimension three, therefore it could have subalgebras of dimension three, two, one, or zero(trivial).

Proposition 3.3 *The Lie algebra $so(3)$ doesn't have a subalgebra of dimension two.*

proof: *Let V be a subalgebra of $so(3)$ of dimension 2. Then there exists vectors $u, v \in V$, where u, v are independent. But $u \times v \in V$, and $u, v, u \times v$ are independent, hence V must be of dimension three. Therefore, we have a contradiction, and V cannot be a subalgebra of dimension two.*

If we let V_1 be a subalgebra of $se(3)$, and we let $\pi(V_1)$ be one dimensional, then all screws of V_1 are of the form $(\lambda x, y)$, where x is fixed. Let $X_1, X_2, \cdots, X_n$ be a basis of V_1. For simplicity, let $X_1 = (x_1; 0)$, $X_2 = (0; y_1)$, $X_3 = (0; y_2)$, and $X_4 = (0; y_3)$. The dimension of V_1 can, at most, be four because $\pi(V_1)$ is of dimension one out of a possibility of three .

It is now known that V_1 is of dimension four or less. The next step is to look at all the possible dimensions for V_1. We will begin with dimension one and proceed to dimension four.

If the dimension of V_1 is one then the basis must be of dimension one. We know $X_1 = (x_1; 0)$, $X_2 = (0; y_1)$, $X_3 = (0; y_2)$, and $X_4 = (0; y_3)$, however, three of these basis vectors must be dependent on the remaining one for V_1 to be of dimension one.

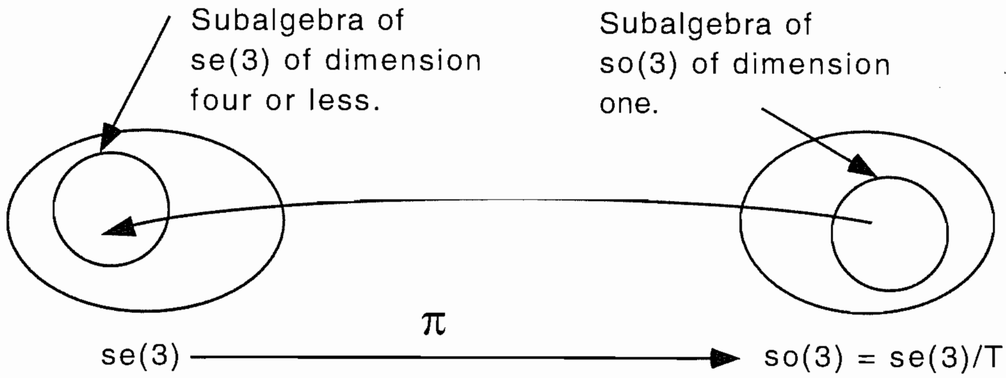


Figure 3.5: The mapping : $se(3) \to so(3) = se(3)/T$.

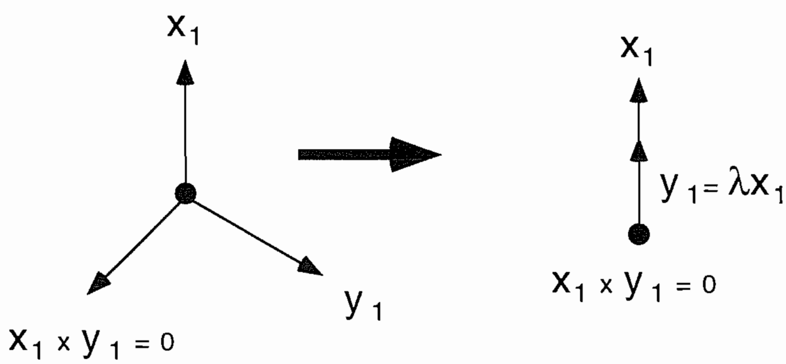


Figure 3.6: The basis of V_1 if $dim(V_1) = 1$ and $dim(\pi(V_1)) = 1$.

Moreover, we know that X_1 cannot be zero because $\pi(V_1) = 1$, therefore X_2. X_3, and X_4 must be dependent on X_1. For simplicity lets assume that $X_3 = X_4 = 0$ and only deal with X_2 and X_1. If X_2 is dependent on X_1, then $X_2 = (0; \lambda x_1)$ where λ is a constant. X_1 and X_2 can be written together in the form $X_1 = (x_1; \lambda x_1)$. Figure 3.6 is a vector diagram that illustrates this point. For this case, if λ is not zero then we have a helicoidal motion along a vector, or if λ is equal to zero then we have a revolute motion. These two cases correspond to two of the lower mechanical joints.

If the dimension of V_1 is two then the the basis must be of dimension two. Therefore, $X_1, \cdots, X_4$ cannot be all independent. For simplicity, let X_3 and X_4 be zero. If X_1 and X_2 are our basis vectors then the cross product between them must be zero for the basis to be of dimension two. Therefore,

$$X_1 \times X_2 = (x_1; y_1) \times (0; y_2) = (0; x_1 \times y_2) = 0 \tag{3.9}$$

$$\Rightarrow x_1 \times y_2 = 0 \rightarrow y_2 = \lambda x_1. \tag{3.10}$$

Hence, the basis is: $X_1 = (x_1; 0)$ and $X_2 = (0; \lambda x_1)$. This case is very similar to the case of dimension one (see Figure 3.6) except that the value of λ is not fixed. Therefore, the rotation in the x_1 direction and the translation in the x_1 direction are independent. This type of motion is called cylindrical motion which is also a lower mechanical joint.

If the dimension of V_1 is three then the basis must also be of dimension three. Therefore, one of the basis vectors $X_1, \cdots, X_4$ is dependent on the other three. Moreover, the cross products between the basis vectors must also be dependent. Let X_4 be the dependent basis vector. Now we know that $X_4 = (0; y_3)$, $X_1 \times X_2 = (0; x_1 \times y_2)$, and $X_1 \times X_3 = (0; x_1 \times y_2)$ must be dependent on X_1, X_2, and X_3. This will be the case if $X_3 = a(X_1 \times X_2)$ where a is a constant, $X_4 = (0; \lambda x_1)$ where λ is a constant, and $X_2 = c(X_1 \times X_3)$ where c is a constant (see Figure 3.7). We can write this basis for this case as $X_1 = (x_1; \lambda x_1)$, $X_2 = (0; y_1)$, and $X_3 = (0; y_2)$. This represents two subgroup classes. If λ is zero then we have planar motion, a lower mechanical pair. If

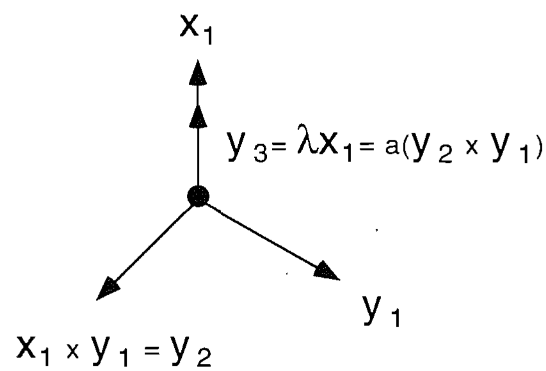


Figure 3.7: The basis of V_1 if $dim(V_1) = 3$ and $dim(\pi(V_1)) = 1$.

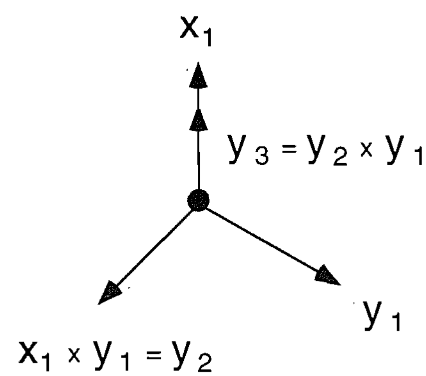


Figure 3.8: The basis of V_1 if $dim(V_1) = 4$ and $dim(\pi(V_1)) = 1$.

λ is not zero then we have planar translation with a helicoidal motion perpendicular to the planar translation, we will call this a "Y-movement."

If the dimension of V_1 is four then the basis must also be of dimension four. Hence the basis vectors $X_1, \cdots, X_4$ must be all independent, however, the cross product between them should be dependent. This result leads to another subgroup classification that we will call "X-movement." This subgroup class corresponds to general translation and one axis rotation (see Figure 3.8). Note that the vectors X_1 and X_2 do not need to line up for this class of group, but the basis is easier to visualize if they are.

This takes care of the subgroup class associated with $\pi(V_1)$ being of dimension one. Now let $\pi(V_1)$ be of dimension three. In other words let $\pi(V_1) = so(3)$. Then the basis can be written in the canonical form $X_1 = (x_1; 0)$, $X_2 = (x_2; 0)$, $X_3 = (x_3; 0)$, $X_4 = (0; y_1)$, $X_5 = (0; y_2)$, and $X_6 = (0; y_3)$.

Proposition 3.4 *There is only one case of dimension four or greater for V_1 given the basis $X_1 = (x_1; 0)$, $X_2 = (x_2; 0)$, $X_3 = (x_3; 0)$, $X_4 = (0; y_1)$, $X_5 = (0; y_2)$, and $X_6 = (0; y_3)$ and $\pi(V_1) = so(3)$. It is the Euclidean group of dimension six.*

proof: *Let the dimension of V_1 be four, then the basis can be written as $X_1 = (x_1; 0)$, $X_2 = (x_2; 0)$, $X_3 = (x_3; 0)$, $X_4 = (0; y_1)$. The first three basis vectors make $se(3)$ and the last basis vector is for the fourth dimension and corresponds to a translation. For V_1 to be of dimension four the cross product between the basis vectors must be dependent. $X_1 \times X_4 = (0; x_1 \times y_1)$, and $X_2 \times X_4 = (0; x_2 \times y_1)$. The result of the two cross products must be independent of the basis vectors and each other, therefore, V_1 is of dimension six, which is $se(3)$. The same result occurs if a basis of five independent basis vectors is used. Hence, the proposition is true.*

From Proposition 3.4 we know that there is only one case of dimension four or greater for V_1 given the basis $X_1, \cdots, X_6$ and $\pi(V_1) = so(3)$; it is the Euclidean group of dimension six. If the dimension of three is considered then the canonical basis would be $X_1 = (x_1; 0)$, $X_2 = (x_2; 0)$, and $X_3 = (x_3; 0)$. This is the basis for the subalgebra $so(3)$, therefore V_1 for $\pi(V_1) = 0$ where V_1 is of dimension three corresponds to the subgroup class $SO(3)$. This is the class of spherical rotations, a lower mechanical joint.

If $\pi(V_1)$ is of dimension zero, then we have a eliminated rotations from the Euclidean group. This only leaves translations. The basis of V_1 in canonical form for this case is $X_1 = (0; y_1)$, $X_2 = (0; y_2)$, and $X_3 = (0; y_3)$. This basis corresponds to general translation if the three basis vectors are independent. If only two of the basis vectors are independent, then V_1 is of dimension two, and we have planar translation. We do not have to worry about the cross product between X_1 and X_2 being independent for this case because the cross product between two translations is zero. If only one basis vector is independent then V_1 is of dimension one, and we have rectilinear translation. This case corresponds to rectilinear motion, a lower mechanical joint – prismatic joint.

This only leaves the trivial case of dimension zero which corresponds to no motion at all.

3.3.3 Classification of Subgroups

Table 3.1 lists all of the continuous subgroup classes for the special Euclidean group. The table also gives the dimension, the notation, and the associated lower pair (assuming one exists) for each subgroup class. These subgroups will be used extensively throughout the rest of this dissertation. It should be noted that the lower case variables used in the notion represent the location of the action of the subgroups.

Classes of Subgroups of the Displacement Group			
Constraint Description	Notation	Degree of Freedom	associated lower pair
identity element	$\{I\}$	0	none
rectilinear translation	$\{T_u\}$	1	(P) prismatic
rotation about an axis	$\{R_u\}$	1	(R) revolute
helicoidal motion	$\{H_{u,p}\}$	1	screw
planar translation	$\{T_P\}$	2	none
cylindrical motion	$\{C_u\}$	2	(C) cylindrical
spatial translation	$\{T\}$	3	none
planar motion	$\{G_P\}$	3	plane
spherical motion	$\{S_o\}$	3	(S) spherical
Y movement	$\{Y_{u,p}\}$	3	none
X movement	$\{X_u\}$	4	none
general motion	$\{D\}$	6	none

Table 3.1: Subgroups of the Euclidean group – displacement group.

The lower case u in $\{T_u\}$, $\{R_u\}$, $\{H_{u,p}\}$, $\{C_u\}$, $\{Y_{u,p}\}$, and $\{X_u\}$ represents the translation, rotation, or helicoidal motion along the axis u. The lower case p used in $\{H_{u,p}\}$ and $\{Y_{u,p}\}$ is the pitch for a helicoidal motion along the axis u. The lower case o used in $\{S_o\}$ is the center of rotation for spherical motion. The upper case P used in $\{T_P\}$ and $\{G_P\}$ is the plane of action for general planar motion and for planar translation.

3.4 Analysis Tools for Fixture Design

In this section, we develop a formal theory for design evaluation of touch sensitive fixtures. The idea being that if we know the location of a set of surfaces in $SE(3)$, we may be able to uniquely determine the location of a frame attached to these surfaces using the geometry of each surface as a guide. The location of each surface would be determined by a set of touches to the surface. For example, it is known that four points to a sphere of unknown radius will uniquely determine the location of that sphere [10]. In Chapter 4, the actual touch contacts to the surfaces are analyzed, however, for this chapter, only surfaces and combinations of surfaces are considered in the design of referencing fixtures. To analyze a set of surfaces, three propositions are provided and proven to be true. These propositions provide the basis for the design procedure developed in Section 3.5.

Definition 3.9 *A* **primitive surface** *of a solid is defined as an algebraic surface that locally coincides with a bounded face of the solid. The primitive features of a cube, for example, are the six infinite planes that bound the solid volume.*

The reason for treating a surface as a primitive surface is understandable when you consider that a set of touches is being made to the surface in order to find its location in space. It would be very difficult for a robot to touch the edge of a surface using a touch sensing probe because the edge has no thickness. Therefore by treating the surface as infinite, the edges do not become involved.

Proposition 3.5 *Let S be a set of primitive surfaces and let G be the symmetry group for the set S. If G contains all of possible rotation elements about an axis, L-L', then the set S cannot be used to uniquely determine the relative location of the frame associated with S to the frame of the touch sensor in three dimensional Euclidean space.*

Proof: *Assume that S can uniquely determine the relative location of the frame associated with S to the robot's frame in three dimensional Euclidean space. Rotate S about the axis L-L' more than zero degrees but less one complete revolution. Since all rotations about L-L' are in the group G, then the set S after the rotation will "look" the same as it did prior to the rotation. However, the frame associated with the set S will no longer be the same frame as it was prior to the rotation. Therefore, the set S cannot be used to uniquely determine the relative location of the frame associated with S to the frame of the sensor in three dimensional Euclidean space.*

Proposition 3.6 *Let S be a set of primitive surfaces and let G be the symmetry group of the set S. If G contains any translations then the set S cannot be used to uniquely determine the relative location of the frame associated with S to the frame of the sensor in three dimensional Euclidean space.*

Proof: *Assume that S can uniquely determine the relative location of the frame associated with S to the frame of the sensor in three dimensional Euclidean space. Translate S using any element translation element in G. Since the translation element is in the group G, then the set S after the translation will "look" the same as it did prior to the translation However, the frame associated with the set S will no longer be the same frame as it was prior to the translation. Therefore, the set S cannot be used to uniquely determine the relative location of the frame associated with S to the frame of the sensor in three dimensional Euclidean space.*

Proposition 3.7 *Let S_1 and S_2 be two sets of primitive surfaces, and let G_1 and G_2 be the symmetry groups for S_1 and S_2. Let S_3 represent the combination of S_1 and S_2, and let G_3 represent the symmetry group associated with S_3. All finite symmetries in G_3 were also finite symmetries in either G_1 or G_2. No new finite symmetries can be created from the combination of surfaces.*

Proof: *It has known that the combination of two symmetry groups results in a group that is either equal in size to the intersection of the two original groups or smaller. Therefore, the new group has no new elements in it that were not in the original groups. Therefore, the only way to get new finite symmetries is by the intersection of continuous groups. The only way to get an intersection of two continuous groups that is not the identity element is by having the continuous groups be the same, resulting in another continuous group. Therefore, no new finite symmetries are created.*

Propositions 3.5, 3.6,and 3.7 are powerful tools for the analysis of any geometric referencing fixture. In most cases a referencing fixture's primitive surfaces can be represented using the simple group notation introduced in the previous section. If it is possible to represent the primitive surfaces using the group notation then the complete fixture can be analyzed by taking the intersections of the group representations of the primitive surfaces. Let $G_1, \cdots, G_n$ represent the group notation for n primitive surfaces that form a referencing fixture. The fixture is a "useful" fixture if:

$$G_1 \cap G_2 \cap G_3 \cap \cdots \cap G_n = \{I\} \tag{3.11}$$

where "useful" means that it can uniquely determine the relative position between the reference frame and the robot end effector. Equation 3.11 is an extension of one of Hervé's equations that he used for the analysis of mechanisms [15].

Equation 3.11 is a very powerful tool, however, the mathematical intersection of two or more groups usually requires some geometric insight that equation 3.11 cannot provide. In addition, equation 3.11 will not always give perfect results for an actual fixture when it comes to finite symmetries. This is due to the fact that the actual fixture may not have finite symmetries that the primitive surface model does have. This may cause a "useful" fixture not to pass equation 3.11 because of the remainder of finite symmetries after the intersection of all group representations. It is, in general, a good idea to use both the propositions and equation 3.11 when analyzing a fixture to be sure that the fixture will work. The use of both methods is discussed in the next section.

3.5 Application of Analysis Tools

3.5.1 Introduction

Once a fixture is broken down into a primitive surface or group of primitive surfaces, then it should not contain any continuous rotation or translation groups. Propositions 3.5 and 3.6 are used to check for these continuous groups. If the fixture does have continuous groups then it cannot be used to uniquely determine the relative

Group Notation for Primitive Surfaces	
Primitive Surface	Notation
Sphere	$\{S_o\}$ where o is the center of the sphere.
Right Cylinder	$\{C_u\}\{rot(v, n\pi)\}$ where u is the axis of the cylinder, $n \in N$, and $v \perp u$.
Plane	$\{G_P\}\{rot(v, n\pi)\}$ where v is in the plane P and $n \in N$.

Table 3.2: Group notation for a sphere, plane, and right cylinder.

location of the fixture to the tool. Finally, finite symmetries must be checked for their effect on the design of the fixture.

Using Proposition 3.7, it is known that all finite symmetries in the final fixture originate from each individual surface. Therefore, a possible and useful way to check for finite symmetries is to look at the finite symmetries of each surfaces of a fixture on a one-by-one basis. Among the sphere, cylinder, and the plane[1] only the sphere has no finite symmetries which makes it easy to work with (Figure 3.9a). The cylinder, when treated as a primitive surface, has an infinite number of finite symmetries. Every axis perpendicular to the center line and intersecting the center line of the cylinder has a finite symmetry about it (Figure 3.9b). The plane, when treated as a primitive surface, also has an infinite number of finite symmetries. Any axis through the plane has a finite rotational symmetry about it (Figure 3.9c). Table 3.2 shows the continuous and finite group notation for the sphere, plane, and right cylinder.

After breaking the fixture down into individual primitive surfaces and knowing

[1]For simplicity, spheres, planes, right cylinders, and their combinations will be used as examples to demonstrate the application of the propositions and equation 3.11, however in Section 3.6 other surfaces will be discussed.

the finite symmetries of these primitive surfaces, it is time to see if the finite symmetries are still present after the addition of the other primitive surfaces to the fixture. Each primitive surface should be judged relative to the other surfaces to see if the finite symmetries go away. If there are no finite symmetries left, then the fixture is "useful." If the fixture still has some finite symmetries left, then the real shape of the fixture(not the combination of the primitive surfaces) may or may not eliminate the finite symmetry. For example, a fixture may have a planar surface that can only be reached on one side, this eliminates the finite rotation of the primitive planar surface associated with the real planar surface (Note: the group representation would simply be $\{G_P\}$ for this case). If there are finite symmetries left after completely analyzing the fixture then the fixture will not uniquely determine the location of the fixture to the sensor – it will not be "useful." This is similar to getting the result $\{I\}$ using equation 3.11. Several examples are given to better illustrate this step.

Of the three surfaces being used for the example, the sphere, cylinder, and plane, none can be used by themselves to uniquely determine the relative position of the fixture frame to the frame of the sensor in $SE(3)$. Therefore, a combination of these surfaces must be used to make a proper fixture. However, it is useful to show the problems with each of these surfaces when used alone.

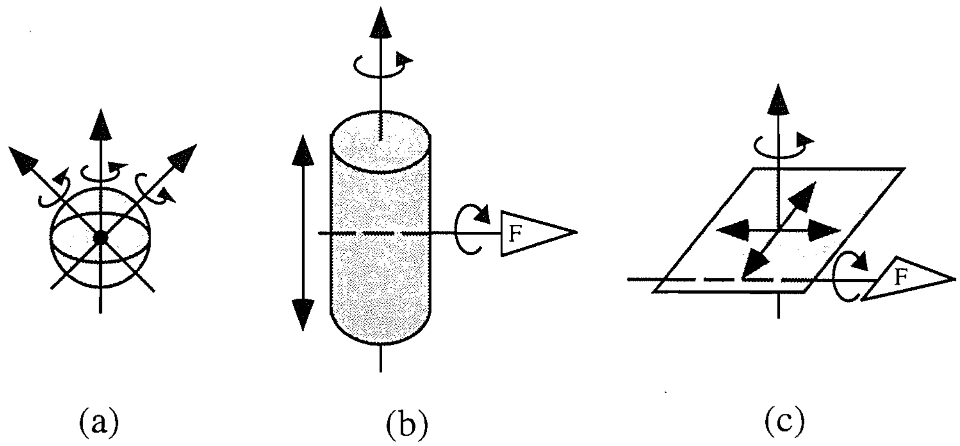


Figure 3.9: (a) A single sphere fixture, (b) A single cylinder fixture, (c) A single plane fixture.

3.5.2 Examples: Basic Surfaces

A sphere (Figure 3.9a) has no finite rotational symmetries and no continuous translational symmetries, however, it does have an infinite number of continuous rotational symmetries. Any axis through the center of the sphere can be used to create a continuous rotational symmetry. Obviously, one sphere can not be used for a complete fixture. This can all be seen in the group notation for the sphere $\{S_o\}$.

A cylinder (Figure 3.9b) has finite rotational symmetries, a continuous rotational symmetry, and a continuous translational symmetry. The continuous translational symmetry comes from the fact that the cylinder, treated as a primitive surface, can be translated in the direction of the center line of the cylinder and the cylinder will look the same. The continuous rotational symmetry comes from the fact that any rotation about the center line of the cylinder returns the cylinder to itself. The finite rotational symmetries come from flipping the cylinder on an axis perpendicular to the center line of the cylinder. Because the cylinder is treated as a primitive surface, the cylinder when rotated 180 degrees returns to itself. This can all be seen in the group notation for the cylinder $\{C_u\}\{rot(v, n\pi)\}$ where u is the axis of the cylinder, $n \in N$, and $v \perp u$.

A plane (Figure 3.9c) has finite rotational symmetries, continuous rotational symmetries, and continuous translational symmetries. The translational symmetries are due to the fact that any movement of the plane in a direction contained in the plane, returns the plane to itself. The finite rotational symmetries, like the cylinder, are 180 degree flipping symmetries. The continuous rotational symmetries come from any rotation about an axis perpendicular to the plane. When the plane is rotated by any of these axis's, the plane returns to itself. This can all be seen in the group notation for the plane $\{G_P\}\{rot(v, n\pi)\}$ where v is in the plane P and $n \in N$.

3.5.3 Examples: Combinations of Surfaces

Now that the basic surfaces have been covered, combinations of these surfaces should be judged for the usefulness in a fixture design. The simplest combinations to start with are combinations of spheres because spheres do not have finite rotational symmetries. A fixture containing two spheres, Figure 3.10a, still will not be a complete fixture because a continuous rotational symmetry exists. The axis for this symmetry is through the center of both spheres. If three spheres are used, Figure 3.10b, a complete fixture will exist as long as the centers of the three spheres are non-collinear. If the

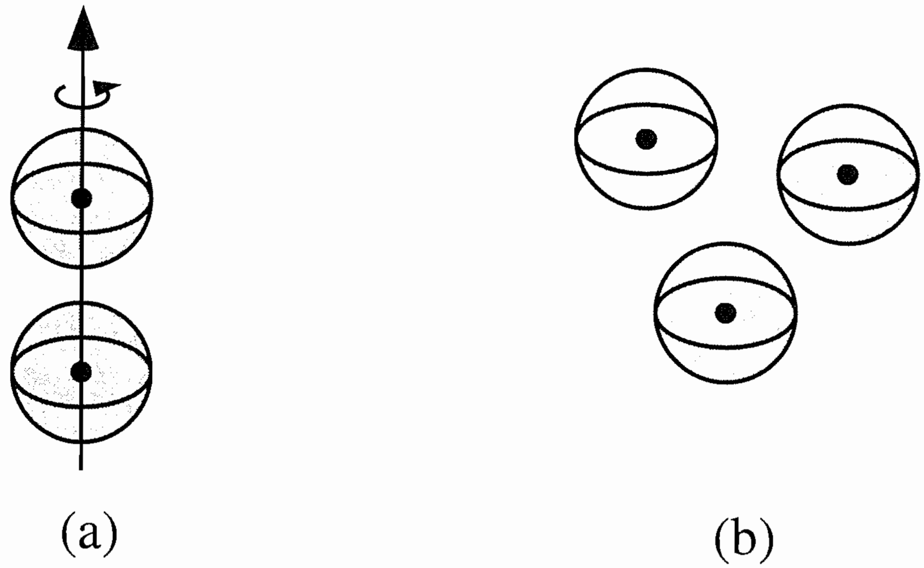


Figure 3.10: (a) A two sphere fixture (b) A three sphere fixture

spheres are collinear, a continuous rotational symmetry through the center of the three spheres exists. From equation 3.11 the combination of two spheres results in

$$\{S_o\} \cap \{S_{o'}\} = \{R_u\} \tag{3.12}$$

where u is the axis through centers o and o'. The combination of three spheres results in

$$\{S_o\} \cap \{S_{o'}\} \cap \{S_{o''}\} = \{I\}. \tag{3.13}$$

If the centers of the spheres are collinear then the result will again be $\{R_u\}$ where u is now the axis through all three sphere centers.

Fixtures containing just planes are a little more difficult to judge than spheres because they may contain finite rotation groups. A fixture containing just two planes will not be a "useful" fixture because there will be continuous groups for any configuration of two planes in addition to finite symmetries (Figure 3.11a). If the two planes intersect then a continuous translational group exists in the direction of the line formed by the intersection of the two planes. If the planes are parallel then there are rotational and translational continuous groups in the same directions as the one plane case. If the two planes are perpendicular (see Figure 3.11) notice that there are two finite symmetries that are created in addition to the finite symmetry along the line of intersection. These two finite symmetries can be eliminated by making the two planes intersect at an angle other than 90 degrees. From equation 3.11 the combination of two planes results in

$$\{G_P\}\{rot(v,n\pi)\} \cap \{G_{P'}\}\{rot(w,n\pi)\} = \{T_u\}\{rot(u,n\pi)\} \qquad (3.14)$$

where u is the line created by the intersection of the two planes. Equation 3.14 is valid when the angle between the planes is not zero nor 90 degrees. If the angle of intersection is 90 degrees then the result contains two more finite symmetries. If the planes are parallel then the result is general planar motion without the finite symmetries. If the two planes are coincident then nothing is obtained from the combination of the two planes.

If three planes are used and none of the planes or lines formed by the intersection of the planes are parallel to each other then no continuous groups exist for that fixture (Figure 3.11b). However, finite groups can exist. They will exist when any of the planes are perpendicular to any of the other planes. When all three planes are mutually perpendicular, the fixture has many finite symmetries. These finite symme-

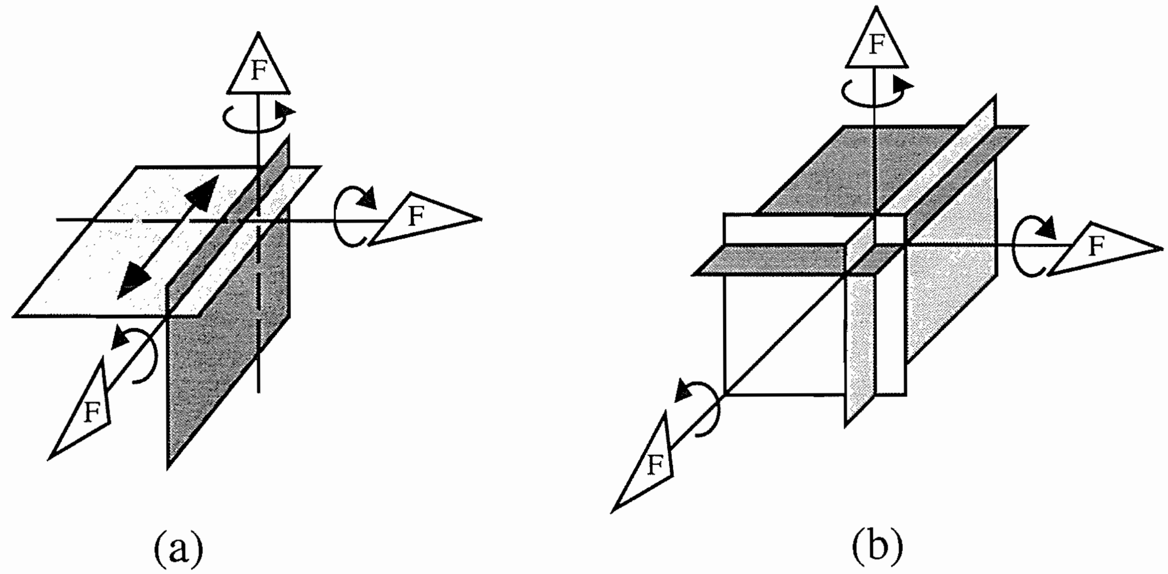


Figure 3.11: (a) A two plane fixture (b) A three plane fixture

tries can be eliminated by the design of the real fixture. If the fixture's sides do not extend past the edges of the other sides then there will be no finite symmetries. This is the cubical fixture design used by McCallion and Pham [23]. From equation 3.11 the combination of three planes results in

$$\{G_P\}\{rot(v, n\pi)\} \cap \{G_{P'}\}\{rot(w, n\pi)\} \cap \{G_{P''}\}\{rot(x, n\pi)\} = \{I\}. \tag{3.15}$$

Equation 3.15 is valid when none of the planes are perpendicular nor parallel to each other.

Spheres and planes can be used together to form fixtures. If one plane is used with one sphere to from a fixture, that fixture will have a continuous rotational group about the axis through the center of the sphere and perpendicular to the plane (Figure 3.12a). If the sphere has its center located in the plane a finite symmetry

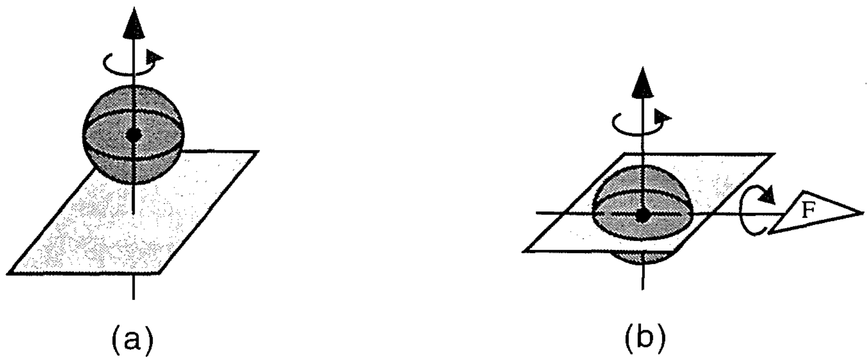


Figure 3.12: (a) A one sphere - one plane fixture (b) A one plane - one sphere fixture with an extra finite symmetry

will exist in addition to the continuous rotation symmetry (see Figure 3.12b). From equation 3.11 the combination of a sphere and plane where the center of the sphere is located outside of the plane results in

$$\{G_P\}\{rot(v, n\pi)\} \cap \{S_o\} = \{R_u\} \tag{3.16}$$

where u goes through o and is perpendicular to P. It is obvious that a one plane, one sphere fixture doesn't satisfy the requirements.

If two planes are used with one sphere, the fixture will not have any continuous groups as long as the two planes are not parallel (Figure 3.13a). If the center of the sphere is located outside of these two planes then the fixture will not have any finite symmetries also. If the sphere is located with its center on one or both planes then finite symmetries may exist. If two spheres are used with one plane to form a fixture, the fixture will not have any symmetry problems as long as at least one sphere is located outside of the plane and the axis through the center of the two spheres is not perpendicular to the plane (Figure 3.13b).

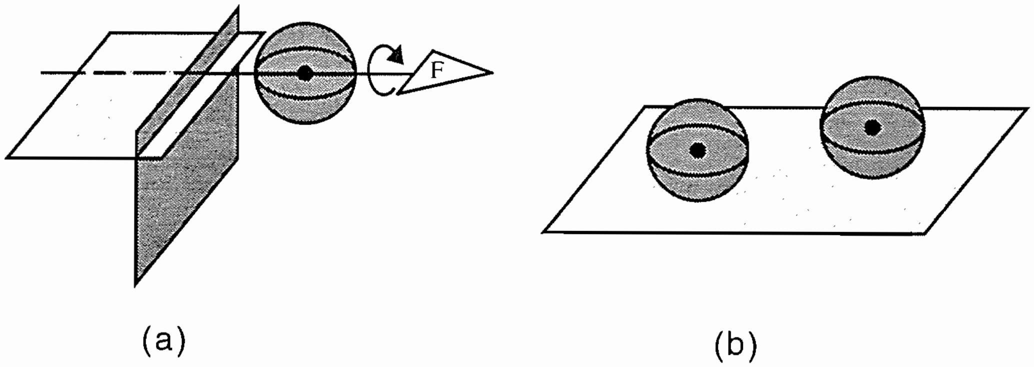


Figure 3.13: (a) A two plane - one sphere fixture (b) A two sphere - one plane fixture

Two cylinders can also be used to form a fixture. If the cylinders are parallel then a continuous translational group will exist in the direction of the center line of both cylinders. However, placing the cylinders at an angle to each other will solve this problem (Figure 3.14). If this is done, finite symmetries may still exist. The relative placement of the two cylinders and the finite length of the cylinders can be used to eliminate this problem in the design of a fixture consisting of two cylinders. From equation 3.11 a two cylinder fixture where the center lines of the cylinders are not parallel result in

$$\{C_u\}\{rot(v, n\pi)\} \cap \{C_{u'}\}\{rot(v, n\pi)\} = \{rot(v, n\pi)\} \tag{3.17}$$

where $v \perp u$ and $v \perp u'$. As stated earlier, the finite symmetry created in the intersection of the two groups can be eliminated by properly designing the actual fixture.

Fixtures can also be made using cylinders and other objects. For example, a sphere-cylinder fixture can be made that will not have any continuous groups as long as the sphere's center is not located on the center line of the cylinder (Figure 3.15). This fixture will, however, always have a finite group associated with it when using

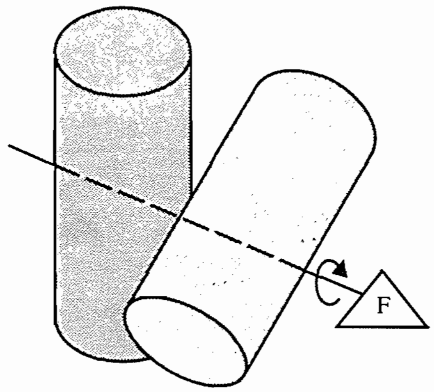


Figure 3.14: A two cylinder fixture

the primitive feature representations for the cylinder and the sphere. The finite symmetry is due to the rotation about an axis through the center of the sphere and perpendicular to the center line of the cylinder. From equation 3.11,

$$\{S_o\} \cap \{C_u\}\{rot(v, n\pi)\} = \{rot(w, n\pi)\} \tag{3.18}$$

where $w \perp u$ and w goes through o. The actual fixture can be designed to eliminate this finite symmetry by placing the sphere outside of the actual range of the cylinder's center line.

A cylinder can also be combined with a plane to form a useful fixture (Figure 3.16b). If the plane is not parallel nor perpendicular to the center line of the cylinder then there will be no continuous symmetries. Again, there will be a finite symmetry problem, however, the actual fixture will not have this finite symmetry because of mechanical constraints in the design of such a fixture. If the plane is perpendicular to the center line of the cylinder, then there is a continuous rotation symmetry about the center line of the cylinder (Figure 3.16a). However, this rotation

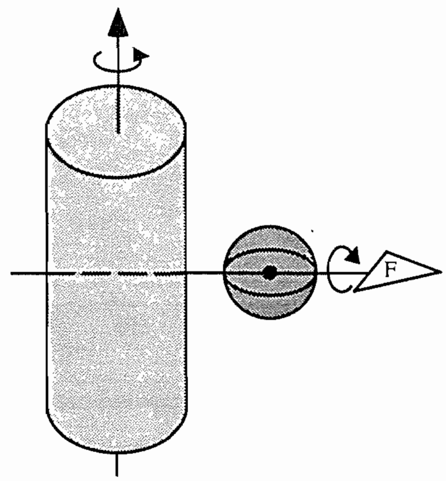


Figure 3.15: A one cylinder - one sphere fixture

symmetry can be eliminated by the addition of another primitive surface.

The actual design of a cylinder-plane fixture where the plane intersects the center line of the cylinder at 45 degrees is discussed in detail in Chapter 5.

Table 3.3 gives an overview of all of the fixtures described in this section including the problems associated with each fixture if there are any. In Section 3.6 other primitive surfaces are analyzed for their use in the design of fixtures.

3.6 Other Surfaces for Use in Fixtures

Although up until now we have only concentrated on fixtures comprising of spheres, planes, cylinders, and combinations, the theory works equally well on all geometric surfaces. Quadratic surfaces, for example, make up a powerful set of geometric surfaces. In fact spheres and right cylinders are specific quadratic surfaces, and planes

Fixture elements	Pass Prop. 3.5?	Pass Prop 3.6?	Condition needed to pass rotation and translation propositions	Can work ?
one sphere	No	Yes		No
two spheres	No	Yes		No
three spheres	Cond	Yes	center of spheres cannot be collinear	Yes
one plane	No	No		No
two planes	Cond	No	planes cannot be parallel	No
three planes	Cond	Cond	planes and the lines formed by intersection of the planes cannot be parallel	Yes
one cylinder	No	No		No
two cylinders	Cond	Cond	center line of cylinders cannot be parallel	Yes
1 plane, 1 sphere	No	Yes		No
2 planes, 1 sphere	Cond	Yes	a line through the center of both spheres cannot be perpendicular to the plane	Yes
1 sphere, 2 planes	Cond	Yes	planes cannot be parallel	Yes
1 cyl, 1 plane	Cond	Cond	the plane cannot be parallel nor perpendicular to the center line of the cylinder	Yes
1 cyl, 1 sphere	Cond	Yes	the sphere's center cannot be located on the cylinder's center line	Yes

Table 3.3: Results of analysis using propositions on spheres, cylinders, planes, and combinations of these elements.

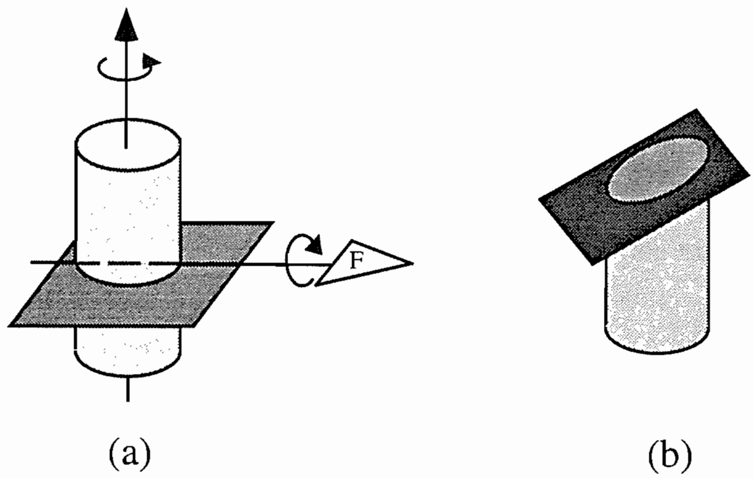


Figure 3.16: A one cylinder - one plane fixture

can be written in a quadratic form. The standard form for a quadratic surface [1] is

$$X^2 + aY^2 + bZ^2 + cXY + dXZ + eYZ + fX + gY + hZ + j = 0. \tag{3.19}$$

There are several quadratic surfaces that do not have any continuous symmetries describing them. These surfaces have the advantage that they could be used alone. Three examples that meet this criteria are an elliptic cone, an ellipsoid, and the hyperboloid of one sheet.

An elliptic cone (Figure 3.17a) has only finite symmetries and can be described using the following group notation: $\{rot(x, l\pi)\}\{rot(y, m\pi)\}\{rot(z, n\pi)\}$ where z is coincident with the center line of the cone, y goes through the cone singularity and in the direction of the major axis of the cone, x goes through the cone singularity and in the direction of the minor axis of the cone, and $l, m, n \in N$. A special case of the elliptic cone occurs when the cone is circular. For this case the cone does have a continuous rotation symmetry about its center. The group notation for this case is $\{R_z\}\{rot(y, m\pi)\}$ where z is coincident with the center of the cone and y is perpendicular to z and runs through the singular point of the cone. The equation for a general cone [39] is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0. \tag{3.20}$$

For a circular cone, $a = b$ in equation 3.20.

An ellipsoid (Figure 3.17b) also has only finite symmetries and can be described using the same group notation: $\{rot(x, l\pi)\}\{rot(y, m\pi)\}\{rot(z, n\pi)\}$ where x, y, and z are the axes of the ellipsoid and $l, m, n \in N$. The equation for a general ellipsoid [39] is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{3.21}$$

Several special cases exist for the ellipsoid. If, using equation 3.21, $a = b$ and $a > c$ then the ellipsoid is called a oblate spheroid (the ellipsoid looks like a door knob), and it has a continuous rotation symmetry about axis z. If $a = b$ and $a < c$, then the ellipsoid is called a prolate spheroid (the ellipsoid looks like an egg), and it also has a continuous rotation symmetry about axis z. Finally, if $a = b = c$, then the ellipsoid is a sphere of radius a.

A hyperboloid of one sheet (Figure 3.17c) has the same properties as an elliptic cone. It has the equation [39]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{3.22}$$

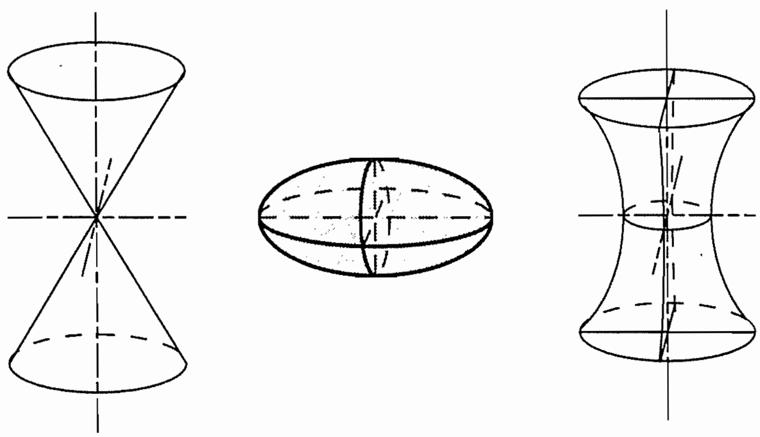


Figure 3.17: Some other quadratic surfaces.

Chapter 4

Contact Analysis

In the previous chapter, reference fixtures were analyzed based on the geometric surfaces that composed them. A reference fixture was considered "useful" if a fixture could be located in space using the properties of its surfaces. For example, a three sphere fixture was considered "useful" because the center of each sphere could be used to locate the fixture in space. The location of a fixture's surfaces was assumed to be determinable, however, no methods for finding the surface locations was described.

In this chapter, the determination of the location of a fixture's surfaces is considered in the design of the fixture. The determination of the location is based upon coincidence relations between geometric elements caused by contact. For example, a robotic touch sensor that touches a plane would be considered a plane-point contact. Given a set of geometric contacts between a fixture and a probe, a fixture can be analyzed to see if the fixture can be located in space. Figure 4.1 illustrates the contact

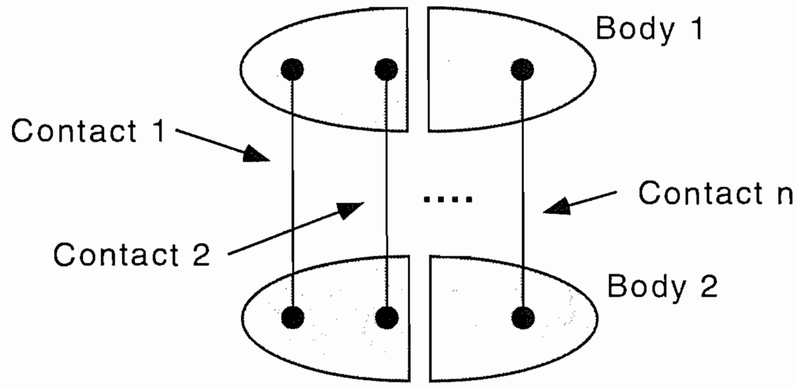


Figure 4.1: Contacts between two bodies.

between the fixture, body 2, and the probe, body 1. The end result of this analysis is the enumeration of all possible fixtures involving a set contacts.

The enumeration of fixture designs using contact analysis requires that all possible contacts are known. Only contacts between points, lines, spheres, planes, and cylinders are being considered. In Section 4.1, all possible contacts between these elements are listed. In Section 4.2, using group theory, the group representation for each contact is found. In Section 4.3, combinations of the contacts are analyzed to determine if a fixture making these contacts is "useful." The analysis of the combination of contacts uses group theory and techniques developed by Hervé [15] for mobility analysis of mechanisms. Much of the background can be found in [15], [30], and [31].

Section 4.3 concludes with the listing of all possible "useful" contact combinations for the design of mechanical fixtures. Section 4.4 describes more practical aspects of the enumeration. Finally, Section 4.5 describes and gives algebraic methods and examples for calculation of surface locations needed for implementation of this fixture analysis technique. It should be noted that although the scope of this dissertation is limited to contacts involving five specific geometric contacts, the techniques developed are general and can be applied to other geometric features and non tactile sensing elements used in robotic referencing and calibration.

4.1 List of Geometric Element Contacts

In order to analyze combinations of geometric elements, it is essential to know all of the possible contacts existing between the geometric elements being considered. Therefore, every possible contact between a line, point, sphere, plane, and cylinder must be considered. Moreover, there are two types of contacts to to be considered for every geometric element, fixed and mobile.

If the position of the contact is known in the local coordinate system of the geometric element, we shall refer to the contact as a fixed contact (F). Otherwise, the contact is called a mobile contact (M). For example, if a point comes into contact with a planar surface, the contact is considered fixed if the location of the contact is known in the surface's frame and mobile otherwise. Note, a point is always a fixed contact because the location of a touch to its body must be the point itself. Using points, spherical surfaces, and planar surfaces, cylindrical surfaces, and lines with both mobile and fixed contacts, a list of all possible contacts is shown in Table 4.1.

In all of the cases studied here, it is assumed that the geometric elements are

List of Possible Contacts		
Objects in Contact	Type of Contact	Abbreviated Notation for Contact
Point - Point	(F/M) - (F/M)	P/F - P/F
Point - Sphere	(F/M) - F	P/F - S/F
Point - Sphere	(F/M) - M	P/F - S/M
Point - Plane	(F/M) - F	P/F - PL/F
Point - Plane	(F/M) - M	P/F - PL/M
Point - Line	(F/M) - F	P/F - L/F
Point - Line	(F/M) - M	P/F - L/M
Point - Cylinder	(F/M) - F	P/F - C/F
Point - Cylinder	(F/M) - M	P/F - C/M
Sphere - Sphere	F - F	S/F - S/F
Sphere - Sphere	F - M	S/F - S/M
Sphere - Sphere	M - M	S/M - S/M
Sphere - Plane	F - F	S/F - PL/F
Sphere - Plane	F - M	S/F - PL/M
Sphere - Plane	M - F	S/M - PL/F
Sphere - Plane	M - M	S/M - PL/M
Sphere - Line	F - F	S/F - L/F
Sphere - Line	F - M	S/F - L/M
Sphere - Line	M - F	S/M - L/F
Sphere - Line	M - M	S/M - L/M
Sphere - Cylinder	F - F	S/F - C/F
Sphere - Cylinder	F - M	S/F - C/M
Sphere - Cylinder	M - F	S/M - C/F
Sphere - Cylinder	M - M	S/M - C/M
Plane - Plane	(F/M) - (F/M)	PL/F - PL/F
Plane - Line	F - F	PL/F - L/F
Plane - Line	M - F	PL/M - L/F
Plane - Cylinder	F - F	PL/F - C/F
Plane - Cylinder	F - M	PL/F - C/M
Plane - Cylinder	M - F	PL/M - C/F
Plane - Cylinder	M - M	PL/M - C/M
Line - Line	F - F	L/F - L/F
Line - Line	F - M	L/F - L/M
Line - Line	M - M	L/M - L/M
Line - Cylinder	F - F	L/F - C/F
Line - Cylinder	F - M	L/F - C/M
Line - Cylinder	M - F	L/M - C/F
Line - Cylinder	M - M	L/M - C/M
Cylinder - Cylinder	F - F	C/F - C/F
Cylinder - Cylinder	F - M	C/F - C/M
Cylinder - Cylinder	M - M	C/M - C/M
KEY: F = fixed, M = mobile, P = point, L = line, S = sphere, PL = plane, C = cylinder		

Table 4.1: Possible contacts between lines, spheres, planes, points, and cylinders.

infinite (in other words, primitive elements). For example, a line would not have end points and a plane would not have edges. This means that if a line is in contact with a plane, it is lying in the plane, not intersecting it. This will be the case throughout this dissertation.

4.2 Group Representation of Geometric Contacts

In this section, we develop a method for evaluation of the contacts listed in Table 4.1. The method used involves the use of the Euclidean group and its subgroups (see Chapter 3 for information on the Euclidean group). Each of the geometric element contacts is transformed into an equivalent group representation. In Section 4.3, these group representations will be used to analyze combinations of contacts for their usefulness in measuring the relative position between two bodies.

The process of finding the group representation for a contact between two geometric elements is a simple one. First the contact should be described using standard joints (e.g., revolute joints, prismatic joints, and spherical joints). Then, each of these joints can be described by their respective subgroups of the Euclidean group (see Chapter 3) . The resulting group representation for a contact is the composition of each of these subgroups.

In many instances, simplifications can be made to the group representation to make it more compact and more understandable. Given a compositions of two subgroups, it may be possible to join the two together and form a larger subgroup. For example, two linear translational groups $\{T_u\}$ and $\{T_{u'}\}$ can sometimes be joined to form one planar translational group, $\{T_u\}\{T_{u'}\} = \{T_P\}$. Moreover, some compositions of groups can be rewritten, resulting in a group of the same dimension. For example, two linear translational groups that are about the same line (line l) or parallel to line l can be composed into one linear translational group about the same line l. The group representation for a point - plane contact and a sphere - sphere contact are given below to further explain the derivation of the group representations.

Given a point - mobile plane contact (P/F-PL/M), the relative motion between the point and the plane can be broken down into two prismatic joints (their lines of action are perpendicular), a revolute joint (the axis is perpendicular to the plane formed by the translational lines), and a spherical joint (the location of the joint is at a point on the plane formed by the two translational lines). The spherical joint represents the point contact with the planar surface and the other components are due to the planar surface being mobile. Figure 4.2 shows a picture of the contact and a schematic of the joint motion associated with that contact. The equation showing the composition of the groups and the resulting group representation for this particular contact is

$$(\{T_d\}\{T_{d'}\})(\{R_u\}\{S_o\}) = \{T_P\}\{S_o\} \tag{4.1}$$

where d is perpendicular to d', u is perpendicular to both d and d', and o lies on the plane formed by d and d'. As can be seen, there are two simplifications performed on the composition in order to make it "cleaner." The dimension of this group repre-

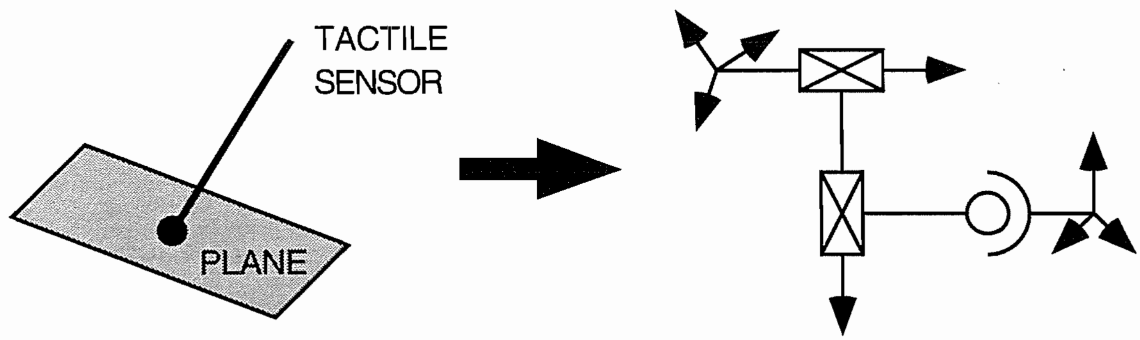


Figure 4.2: A point-mobile planar surface contact.

sentation is five, two from $\{T_P\}$ and three from $\{S_o\}$. Note, if the planar surface was fixed that the resulting motion would be equivalent to just a spherical joint, $\{S_o\}$.

Given a sphere - sphere contact, the relative motion between two spherical surfaces can be broken down into two spherical joints, a single spherical joint, or a revolute joint depending on if the surfaces are mobile, fixed, or one is mobile and one is fixed. A two spherical joint motion will occur if both spherical surfaces are mobile because neither surface will know exactly where the contact occurred. A spherical joint motion will occur if one of the spherical surfaces is mobile and one is fixed because one of the surfaces will know where the contact occurred relative to its coordinate system and the other surface will not. Therefore, the mobile surface can be rotated about any axis through the center of its spherical surface, and no change will be detected by either spherical surface. If both spherical surfaces are fixed then the two surfaces can only rotate about an axis through the center of both spherical surfaces without a change being detected. This results in the equivalent motion of a revolute joint with its axis through the centers of both spherical surfaces. Figure 4.3 illustrated the

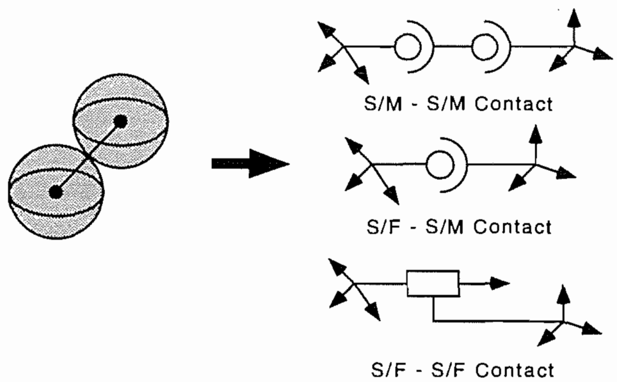


Figure 4.3: A spherical-spherical surface contact.

three possible contacts between a sphere - sphere contact. Note, the case when both surfaces are mobile results in a spherical joint - spherical joint combination. This combination has only a dimension of five, not six. This is caused by a redundant motion in the combination. Both spherical joints can rotate about an axis through the center of both spherical surfaces. Therefore, only one of them is included in the group representation resulting in a decrease in the dimension for the combination.

Table 4.2 lists all of the group representation and dimension for all of the possible contacts. It should be noted that many of the group representations are the same for different contacts.

4.3 Analysis of Contact Combinations

With all of the contacts described by their respective group representation and dimension, it is possible to apply the methods Hervé [15] described to combinations of these contacts to find if they can be used to make a complete measurement of the

Type of Contact	Group Representation	Dim.	Description
P/F - P/F	$\{S_o\}$	3	o is the contact pt.
P/F - S/F	$\{S_o\}$	3	o is the contact pt.
P/F - S/M	$\{S_o\}\{S_{o'}\}$	3+3-1=5	o is the contact pt., o' is the center of S/M
P/F - PL/F	$\{S_o\}$	3	o is the contact pt.
P/F - PL/M	$\{S_o\}\{T_P\}$	3+2=5	P is coincident with PL/M, o is the contact pt.
P/F - L/F	$\{S_o\}$	3	o is the contact pt.
P/F - L/M	$\{S_o\}\{T_u\}$	3+1=4	o is the contact pt., axis u goes through o
P/F - C/F	$\{S_o\}$	3	o is the contact pt.
P/F - C/M	$\{S_o\}\{C_u\}$	3+2=5	o is the contact pt., axis u is a dist. r away from o
S/F - S/F	$\{R_u\}$	1	axis u goes through both sphere centers
S/F - S/M	$\{S_o\}$	3	o is the center of the mobile sphere S/M
S/M - S/M	$\{S_o\}\{S_{o'}\}$	3+3-1=5	o and o' are the centers of the spheres
S/F - PL/F	$\{R_u\}$	1	$u \perp PL/F$ and goes through the center of S/F
S/F - PL/M	$\{G_P\}$	3	P is coincident with PL/M
S/M - PL/F	$\{S_o\}$	3	o is the center of the mobile sphere S/M
S/M - PL/M	$\{S_o\}\{T_P\}$	2+3=5	o is the center of S/M, P is coincident with PL/M
S/F - L/F	$\{R_u\}\{R_{u'}\}$	1+1=2	axis u goes through the center of S/F and the contact pt., axis u' is coincident with L/F
S/F - L/M	$\{R_u\}\{C_{u'}\}$	1+2=3	axis u goes through the center of S/F and the contact pt., axis u' is coincident with L/F
S/M - L/F	$\{S_o\}\{R_u\}$	3+1=4	o is the center of S/M, axis u is coincident with L/F
S/M - L/M	$\{S_o\}\{C_u\}$	3+2=5	o is the center of S/M, axis u is coincident with L/M
S/F - C/F	$\{R_u\}$	1	axis u goes through the center of S/F and the contact pt.
S/F - C/M	$\{R_u\}\{C_{u'}\}$	1+2=3	axis u goes through the center of S/F and the contact pt., and axis u' is coincident with the center line of C/M
S/M - C/F	$\{S_o\}$	3	o is the center of S/M
S/M - C/M	$\{S_o\}\{C_u\}$	3+2=5	o is the center of S/M, and axis u' is coincident with the center line of C/M
PL/F - PL/F	$\{G_P\}$	3	P is coincident with both planes
PL/F - L/F	$\{R_u\}$	1	axis u is coincident with L/F and on the plane
PL/M - L/F	$\{G_P\}\{R_u\}$	3+1=4	P is coincident with PL/M, and axis u is coincident with L/F and on P
PL/F - C/F	$\{T_u\}$	1	axis u is coincident with the center line of C/F
PL/F - C/M	$\{C_u\}$	2	axis u is coincident with the center line of C/M
PL/M - C/F	$\{G_P\}$	3	P is coincident with PL/M
PL/M - C/M	$\{G_P\}\{R_u\}$	3+1=4	axis u is coincident with the center line of C/M and, P is coincident with PL/M
L/F - L/F	$\{S_o\}$	3	o is the contact pt.
L/F - L/M	$\{S_o\}\{T_u\}$	3+1=4	o is the contact pt., and u is coincident with L/M
L/M - L/M	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	2+1+2=5	u is coincident with L/F #1, u'' is coincident with L/M #2, and $u' \perp$ the contact plane.
L/F - C/F	$\{R_u\}\{R_{u'}\}$	1+1=2	u is coincident with L/F, and u' is perpendicular to the contact plane.
L/F - C/M	$\{R_u\}\{R_{u'}\}\{C_{u''}\}$	1+1+2=4	u is coincident with L/F, u'' is coincident with the center line of C/M, and $u' \perp$ the contact plane.
L/M - C/F	$\{R_u\}\{C_{u'}\}$	1+2=3	u' is coincident with L/M, and $u' \perp$ the contact plane.
L/M - C/M	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	2+1+2=5	u is coincident with L/M, u'' is coincident with the center line of C/M, and $u' \perp$ the contact plane.
C/F - C/F	$\{R_u\}$	1	u is perpendicular to the contact plane
C/F - C/M	$\{R_u\}\{C_{u'}\}$	1+2=3	u' is coincident with the center line of C/M, and $u \perp$ the contact plane.
C/M - C/M	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	2+1+2=5	u and u'' are coincident with the center lines of C/M #1 and C/M #2, and $u' \perp$ the contact plane.

Table 4.2: The group representations and dimensions for the geometric contacts.

relative position between two bodies. Hervé stated that if two bodies had multiple constraints between them that the resulting constraint is the intersection of the multiple constraints. Moreover, if the dimension of the resulting intersection is zero, the resulting constraint is fixed and this is what we want to find. Let $\{L_1\}$ through $\{L_n\}$ be the constraints imposed by the geometric element contacts. Then

$$dim(\{L_1\} \cap \{L_2\} \cap \cdots \cap \{L_n\}) = 0. \tag{4.2}$$

In our case, the constraints are given by the group representations. Hence, if two bodies are separated by two point-point contacts then the overall constraint associated with the bodies is the intersection of the two constraints caused by the point-point contacts. It should be noted that there is a difficulty with using the group representations. They are coordinate dependent.

When finding the intersection between two bodies a coordinate system for each of the bodies must be chosen. Most of the group representations assume that the coordinate frames are located in a specific location. For example, any contact with a group representation $\{S_o\}$ assumes that both bodies have their respective coordinate systems located at the center of the spherical joint. Hence, when motion occurs, all displacements are part of the spherical joint group. If two contacts are used with spherical group representations, then only one can have the coordinate systems placed at its center; the other contact will no longer be represented by a spherical joint but a strange set of displacements (see Figure 4.4). Note, changing the coordinate system does not change the dimension of the motion caused by a geometric element contact.

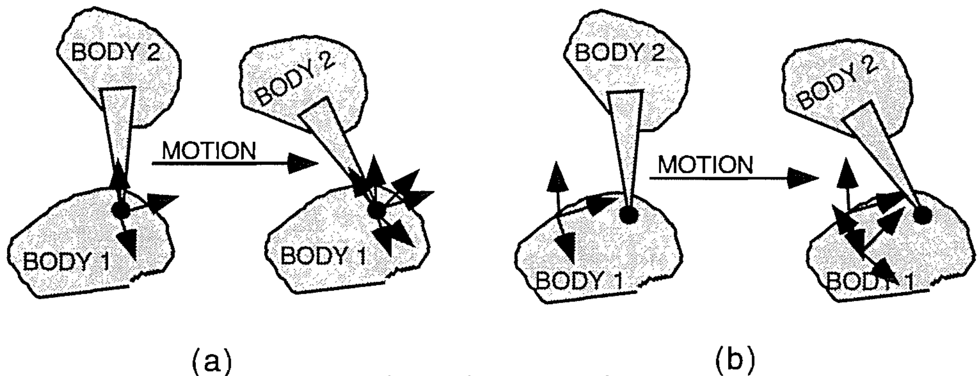


Figure 4.4: Group representation dependence on the location of the coordinate system.

Using equation 4.2, it is now possible to study geometric element combinations and determine if they can be used for measuring the relative position between two bodies. In other words, they can be used to determine if a fixture is "useful." It should be noted that certain geometric element contact configurations have motions in common with other geometric element contact configurations regardless of the locations at which they are placed. For example, When two point-point contacts are used $\{S_o\}$ and $\{S_{o'}\}$, the intersection of their group representations does not result in the identity element as expected, but in a revolute group, $\{R_u\}$, with its axis, u, through the center of both point contacts. Therefore, two point - point contacts does not give a complete solution, and, hence, another piece of information is necessary. A combination of a sphere/fixed - plane/mobile contact with a point - point contact also results in a revolute joint, the axis is through the center of the point and perpendicular to the plane. These cases must be considered when finding complete combinations.

A shortcoming associated with equation 4.2 is that if a combination is found with dimension zero, it still may have a finite number of possible solutions. For example, the theory suggests that it takes three points to find the position of a sphere of known radius. However, if three points are used, two spheres can fit to that one set of points (see Figure 4.10). In order to remedy this situation another piece of information is necessary. This will be the case with many of the combinations found.

Looking at Table 4.2 it is apparent that many of the geometric element contacts share the same group representations. This is due to the fact that a point is a special case of a mobile sphere of radius zero and a line is a cylinder of radius zero. Therefore, all of the possible contacts listed can be reduced from 41 cases to 15. Table 4.3 shows all possible simplifications to the contacts.

Using the notation from Table 4.3, Table 4.4 is a listing of all possible contacts that will result in a set of dimension zero. As stated earlier, a finite number of possible positions may come out of the combination. These cases will require an additional piece of information for a unique solution.

4.4 Point - Surface Contacts

Of the contacts discussed, point-surface contacts are the most common and are currently the most practical for fixture design. The point-surface contacts can be grouped into two classifications, ones involving mobile surfaces and ones involving fixed surfaces.

Classes of Contacts				
The Class Representation	The Group Representation	Contacts in the Class	No. Contacts in Class	Contact Dimension
$\{R\}$	$\{R_u\}$	S/F - S/F, S/F - PL/F, S/F - C/F, PL/F - L/F, C/F - C/F	5	1
$\{T_1\}$	$\{T_u\}$	PL/F - C/F	1	1
$\{R-R\}$	$\{R_u\}\{R_{u'}\}$	S/F - L/F, L/F - C/F	2	2
$\{C\}$	$\{C_u\}$	PL/F - C/M	1	2
$\{R-C\}$	$\{R_u\}\{C_{u'}\}$	S/F - L/M, S/F - C/M, L/M - C/F, C/F - C/M	4	3
$\{S\}$	$\{S_o\}$	P/F - P/F, P/F - S/F, P/F - PL/F, P/F - L/F, P/F - C/F, S/F - S/M, S/M - PL/F, S/M - C/F, L/F-L/F	9	3
$\{G_3\}$	$\{G_P\}$	S/F - PL/M, PL/F - PL/F, PL/M - C/M	3	3
$\{S-R\}$	$\{S_o\}\{R_u\}$	S/M - L/F	1	4
$\{S-T_1\}$	$\{S_o\}\{T_u\}$	P/F - L/M, L/F - L/M	2	4
$\{G_3-R\}$	$\{G_P\}\{R_u\}$	PL/M - L/F, PL/M - C/M	2	4
$\{R-R-C\}$	$\{R_u\}\{R_{u'}\}\{C_{u''}\}$	L/F - C/M	1	4
$\{S-S\}$	$\{S_o\}\{s_{o'}\}$	P/F - S/M, S/M - S/M	2	5
$\{S-C\}$	$\{S_o\}\{C_u\}$	P/F - C/M, S/M - L/M, S/M - C/M	3	5
$\{S-T_2\}$	$\{S_o\}\{T_P\}$	P/F - PL/M, S/M - PL/M	2	5
$\{C-R-C\}$	$\{C_u\}\{R_{u'}\}\{C_{u''}\}$	L/M - L/M, L/M - C/M, C/M - C/M	3	5

Table 4.3: Classes of contacts.

Enumeration of Possible Contact Combinations					
Combination class[1]	No[2]	Combination Class[1]	No[2]	Combination class[1]	No[2]
$\{R\}\{R\}$	15	$\{C\}\{S\}$	9	$\{R-C\}\{S-S\}\{S-C\}\{S-T_2\}$	48
$\{R\}\{T_1\}$	5	$\{C\}\{G_3\}$	3	$\{R-C\}\{S-S\}\{S-C\}\{C-R-C\}$	72
$\{R\}\{R-R\}$	10	$\{C\}\{S-R\}$	1	$\{R-C\}\{S-S\}\{S-T_2\}\{S-T_2\}$	24
$\{R\}\{C\}$	5	$\{C\}\{S-T_1\}$	2	$\{R-C\}\{S-S\}\{S-T_2\}\{C-R-C\}$	48
$\{R\}\{R-C\}$	20	$\{C\}\{G_3-R\}$	2	$\{R-C\}\{S-S\}\{C-R-C\}\{C-R-C\}$	48
$\{R\}\{S\}$	45	$\{C\}\{R-R-C\}$	1	$\{R-C\}\{S-C\}\{S-C\}\{S-C\}$	40
$\{R\}\{G_3\}$	15	$\{C\}\{S-S\}\{S-S\}$	3	$\{R-C\}\{S-C\}\{S-C\}\{S-T_2\}$	48
$\{R\}\{S-R\}$	5	$\{C\}\{S-S\}\{S-C\}$	6	$\{R-C\}\{S-C\}\{S-C\}\{C-R-C\}$	72
$\{R\}\{S-T_1\}$	10	$\{C\}\{S-S\}\{S-T_2\}$	4	$\{R-C\}\{S-C\}\{S-T_2\}\{S-T_2\}$	36
$\{R\}\{G_3-R\}$	10	$\{C\}\{S-S\}\{C-R-C\}$	6	$\{R-C\}\{S-C\}\{S-T_2\}\{C-R-C\}$	72
$\{R\}\{R-R-C\}$	5	$\{C\}\{S-C\}\{S-C\}$	6	$\{R-C\}\{S-C\}\{C-R-C\}\{C-R-C\}$	72
$\{R\}\{S-S\}$	10	$\{C\}\{S-C\}\{S-T_2\}$	6	$\{R-C\}\{S-T_2\}\{S-T_2\}\{S-T_2\}$	16
$\{R\}\{S-C\}$	15	$\{C\}\{S-C\}\{C-R-C\}$	9	$\{R-C\}\{S-T_2\}\{S-T_2\}\{C-R-C\}$	36
$\{R\}\{S-T_2\}$	10	$\{C\}\{S-T_2\}\{S-T_2\}$	3	$\{R-C\}\{S-T_2\}\{C-R-C\}\{C-R-C\}$	48
$\{R\}\{C-R-C\}$	15	$\{C\}\{S-T_2\}\{C-R-C\}$	6	$\{R-C\}\{C-R-C\}\{C-R-C\}\{C-R-C\}$	40
$\{T_1\}\{T_1\}$	1	$\{C\}\{C-R-C\}\{C-R-C\}$	6	$\{S\}\{S\}\{S\}$	165
$\{T_1\}\{R-R\}$	2	$\{R-C\}\{R-C\}$	10	$\{S\}\{S\}\{G_3\}$	135
$\{T_1\}\{C\}$	1	$\{R-C\}\{S\}$	36	$\{S\}\{S\}\{S-R\}$	45
$\{T_1\}\{R-C\}$	4	$\{R-C\}\{G_3\}$	12	$\{S\}\{S\}\{S-T_1\}$	90
$\{T_1\}\{S\}$	9	$\{R-C\}\{S-R\}\{S-R\}$	4	$\{S\}\{S\}\{G_3-R\}$	90
$\{T_1\}\{G_3\}$	3	$\{R-C\}\{S-R\}\{S-T_1\}$	8	$\{S\}\{S\}\{R-R-C\}$	45
$\{T_1\}\{S-R\}$	1	$\{R-C\}\{S-R\}\{G_3-R\}$	8	$\{S\}\{S\}\{S-S\}$	90
$\{T_1\}\{S-T_1\}$	2	$\{R-C\}\{S-R\}\{R-R-C\}$	4	$\{S\}\{S\}\{S-C\}$	135
$\{T_1\}\{G_3-R\}$	2	$\{R-C\}\{S-R\}\{S-S\}$	8	$\{S\}\{S\}\{S-T_2\}$	90
$\{T_1\}\{R-R-C\}$	1	$\{R-C\}\{S-R\}\{S-C\}$	12	$\{S\}\{S\}\{C-R-C\}$	135
$\{T_1\}\{S-S\}$	2	$\{R-C\}\{S-R\}\{S-T_2\}$	8	$\{S\}\{G_3\}\{G_3\}$	54
$\{T_1\}\{S-C\}$	3	$\{R-C\}\{S-R\}\{C-R-C\}$	12	$\{S\}\{G_3\}\{S-R\}$	27
$\{T_1\}\{S-T_2\}$	2	$\{R-C\}\{S-T_1\}\{S-T_1\}$	12	$\{S\}\{G_3\}\{S-T_1\}$	54
$\{T_1\}\{C-R-C\}$	3	$\{R-C\}\{S-T_1\}\{G_3-R\}$	16	$\{S\}\{G_3\}\{G_3-R\}$	54
$\{R-R\}\{R-R\}$	3	$\{R-C\}\{S-T_1\}\{R-R-C\}$	8	$\{S\}\{G_3\}\{R-R-C\}$	27
$\{R-R\}\{C\}$	2	$\{R-C\}\{S-T_1\}\{S-S\}$	16	$\{S\}\{G_3\}\{S-S\}$	54
$\{R-R\}\{R-C\}$	8	$\{R-C\}\{S-T_1\}\{S-C\}$	24	$\{S\}\{G_3\}\{S-C\}$	81
$\{R-R\}\{S\}$	18	$\{R-C\}\{S-T_1\}\{S-T_2\}$	16	$\{S\}\{G_3\}\{S-T_2\}$	54
$\{R-R\}\{G_3\}$	6	$\{R-C\}\{S-T_1\}\{C-R-C\}$	24	$\{S\}\{G_3\}\{C-R-C\}$	81
$\{R-R\}\{S-R\}$	2	$\{R-C\}\{G_3-R\}\{G_3-R\}$	12	$\{S\}\{S-R\}\{S-R\}$	9
$\{R-R\}\{S-T_1\}$	4	$\{R-C\}\{G_3-R\}\{R-R-C\}$	8	$\{S\}\{S-R\}\{S-T_1\}$	18
$\{R-R\}\{G_3-R\}$	4	$\{R-C\}\{G_3-R\}\{S-S\}$	16	$\{S\}\{S-R\}\{G_3-R\}$	18
$\{R-R\}\{R-R-C\}$	2	$\{R-C\}\{G_3-R\}\{S-C\}$	24	$\{S\}\{S-R\}\{R-R-C\}$	9
$\{R-R\}\{S-S\}\{S-S\}$	6	$\{R-C\}\{G_3-R\}\{S-T_2\}$	16	$\{S\}\{S-R\}\{S-S\}$	18
$\{R-R\}\{S-S\}\{S-C\}$	12	$\{R-C\}\{G_3-R\}\{C-R-C\}$	24	$\{S\}\{S-R\}\{S-C\}$	27
$\{R-R\}\{S-S\}\{S-T_2\}$	8	$\{R-C\}\{R-R-C\}\{R-R-C\}$	4	$\{S\}\{S-R\}\{S-T_2\}$	18
$\{R-R\}\{S-S\}\{C-R-C\}$	12	$\{R-C\}\{R-R-C\}\{S-S\}$	8	$\{S\}\{S-R\}\{C-R-C\}$	27
$\{R-R\}\{S-C\}\{S-C\}$	12	$\{R-C\}\{R-R-C\}\{S-C\}$	12	$\{S\}\{S-T_1\}\{S-T_1\}$	27
$\{R-R\}\{S-C\}\{S-T_2\}$	12	$\{R-C\}\{R-R-C\}\{S-T_2\}$	8	$\{S\}\{S-T_1\}\{G_3-R\}$	36
$\{R-R\}\{S-C\}\{C-R-C\}$	18	$\{R-C\}\{R-R-C\}\{C-R-C\}$	12	$\{S\}\{S-T_1\}\{R-R-C\}$	18
$\{R-R\}\{S-T_2\}\{S-T_2\}$	6	$\{R-C\}\{S-S\}\{S-S\}\{S-S\}$	16	$\{S\}\{S-T_1\}\{S-S\}$	36
$\{R-R\}\{S-T_2\}\{C-R-C\}$	12	$\{R-C\}\{S-S\}\{S-S\}\{S-C\}$	36	$\{S\}\{S-T_1\}\{S-C\}$	54
$\{R-R\}\{C-R-C\}\{C-R-C\}$	12	$\{R-C\}\{S-S\}\{S-S\}\{S-T_2\}$	24	$\{S\}\{S-T_1\}\{S-T_2\}$	36
$\{C\}\{C\}$	1	$\{R-C\}\{S-S\}\{S-S\}\{C-R-C\}$	36	$\{S\}\{S-T_1\}\{C-R-C\}$	54
$\{C\}\{R-C\}$	4	$\{R-C\}\{S-S\}\{S-C\}\{S-C\}$	48	$\{S\}\{G_3-R\}\{G_3-R\}$	27
Table Continues on Next Page					

Table 4.4: Enumeration of all possible contact combinations between points, spheres, lines, cylinders, and planes resulting in a dimension of zero.

Enumeration of Possible Contact Combinations (continued)			
Combination class[1]	Number[2]	Combination class[1]	Number[2]
$\{S\}\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}$	18	$\{G_3\}\{S\text{-}T_1\}\{S\text{-}S\}$	12
$\{S\}\{G_3\text{-}R\}\{S\text{-}S\}$	36	$\{G_3\}\{S\text{-}T_1\}\{S\text{-}C\}$	18
$\{S\}\{G_3\text{-}R\}\{S\text{-}C\}$	54	$\{G_3\}\{S\text{-}T_1\}\{S\text{-}T_2\}$	12
$\{S\}\{G_3\text{-}R\}\{S\text{-}T_2\}$	36	$\{G_3\}\{S\text{-}T_1\}\{C\text{-}R\text{-}C\}$	18
$\{S\}\{G_3\text{-}R\}\{C\text{-}R\text{-}C\}$	54	$\{G_3\}\{G_3\text{-}R\}\{G_3\text{-}R\}$	12
$\{S\}\{R\text{-}R\text{-}C\}\{R\text{-}R\text{-}C\}$	9	$\{G_3\}\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}$	6
$\{S\}\{R\text{-}R\text{-}C\}\{S\text{-}S\}$	18	$\{G_3\}\{G_3\text{-}R\}\{S\text{-}S\}$	12
$\{S\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}$	27	$\{G_3\}\{G_3\text{-}R\}\{S\text{-}C\}$	18
$\{S\}\{R\text{-}R\text{-}C\}\{S\text{-}T_2\}$	18	$\{G_3\}\{G_3\text{-}R\}\{S\text{-}T_2\}$	12
$\{S\}\{R\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	27	$\{G_3\}\{G_3\text{-}R\}\{C\text{-}R\text{-}C\}$	18
$\{S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}$	36	$\{G_3\}\{R\text{-}R\text{-}C\}\{R\text{-}R\text{-}C\}$	3
$\{S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}$	81	$\{G_3\}\{R\text{-}R\text{-}C\}\{S\text{-}S\}$	6
$\{S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}$	54	$\{G_3\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}$	9
$\{S\}\{S\text{-}S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	81	$\{G_3\}\{R\text{-}R\text{-}C\}\{S\text{-}T_2\}$	6
$\{S\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}$	108	$\{G_3\}\{R\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	9
$\{S\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}$	108	$\{G_3\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}$	12
$\{S\}\{S\text{-}S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	162	$\{G_3\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}$	27
$\{S\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	54	$\{G_3\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}$	18
$\{S\}\{S\text{-}S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	108	$\{G_3\}\{S\text{-}S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	27
$\{S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	108	$\{G_3\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}$	36
$\{S\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	90	$\{G_3\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}$	36
$\{S\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	108	$\{G_3\}\{S\text{-}S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	54
$\{S\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	162	$\{G_3\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	18
$\{S\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	81	$\{G_3\}\{S\text{-}S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	36
$\{S\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	162	$\{G_3\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	36
$\{S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	162	$\{G_3\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	30
$\{S\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	36	$\{G_3\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	36
$\{S\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	81	$\{G_3\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	54
$\{S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	108	$\{G_3\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	27
$\{S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	90	$\{G_3\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	54
$\{G_3\}\{G_3\}\{G_3\}$	10	$\{G_3\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	54
$\{G_3\}\{G_3\}\{S\text{-}R\}$	6	$\{G_3\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	12
$\{G_3\}\{G_3\}\{S\text{-}T_1\}$	12	$\{G_3\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	27
$\{G_3\}\{G_3\}\{G_3\text{-}R\}$	12	$\{G_3\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	36
$\{G_3\}\{G_3\}\{R\text{-}R\text{-}C\}$	6	$\{G_3\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	30
$\{G_3\}\{G_3\}\{S\text{-}S\}$	12	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}R\}$	1
$\{G_3\}\{G_3\}\{S\text{-}C\}$	18	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}T_1\}$	2
$\{G_3\}\{G_3\}\{S\text{-}T_2\}$	12	$\{S\text{-}R\}\{S\text{-}R\}\{G_3\text{-}R\}$	2
$\{G_3\}\{G_3\}\{C\text{-}R\text{-}C\}$	18	$\{S\text{-}R\}\{S\text{-}R\}\{R\text{-}R\text{-}C\}$	1
$\{G_3\}\{S\text{-}R\}\{S\text{-}R\}$	3	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}$	3
$\{G_3\}\{S\text{-}R\}\{S\text{-}T_1\}$	6	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}$	6
$\{G_3\}\{S\text{-}R\}\{G_3\text{-}R\}$	6	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}S\}\{S\text{-}T_2\}$	4
$\{G_3\}\{S\text{-}R\}\{R\text{-}R\text{-}C\}$	3	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	6
$\{G_3\}\{S\text{-}R\}\{S\text{-}S\}$	6	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}$	6
$\{G_3\}\{S\text{-}R\}\{S\text{-}C\}$	9	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}C\}\{S\text{-}T_2\}$	6
$\{G_3\}\{S\text{-}R\}\{S\text{-}T_2\}$	6	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	9
$\{G_3\}\{S\text{-}R\}\{C\text{-}R\text{-}C\}$	9	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	3
$\{G_3\}\{S\text{-}T_1\}\{S\text{-}T_1\}$	9	$\{S\text{-}R\}\{S\text{-}R\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	6
$\{G_3\}\{S\text{-}T_1\}\{G_3\text{-}R\}$	12	$\{S\text{-}R\}\{S\text{-}R\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	6
$\{G_3\}\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}$	6	$\{S\text{-}R\}\{S\text{-}T_1\}\{S\text{-}T_1\}$	3
Table Continues on Next Page			

Table 4.4: Enumeration of all possible contact combinations between points, spheres, lines, cylinders, and planes resulting in a dimension of zero (continued).

Enumeration of Possible Contact Combinations (continued)

Combination class[1]	Number[2]	Combination class[1]	Number[2]
{S-R}{S-T_1}{G_3-R}	4	{S-R}{S-S}{S-C}{C-R-C}{C-R-C}	36
{S-R}{S-T_1}{R-R-C}	2	{S-R}{S-S}{S-T_2}{S-T_2}{S-T_2}	8
{S-R}{S-T_1}{S-S}{S-S}	6	{S-R}{S-S}{S-T_2}{S-T_2}{C-R-C}	18
{S-R}{S-T_1}{S-S}{S-C}	12	{S-R}{S-S}{S-T_2}{C-R-C}{C-R-C}	24
{S-R}{S-T_1}{S-S}{S-T_2}	8	{S-R}{S-S}{C-R-C}{C-R-C}{C-R-C}	20
{S-R}{S-T_1}{S-S}{C-R-C}	12	{S-R}{S-C}{S-C}{S-C}{S-C}	15
{S-R}{S-T_1}{S-C}{S-C}	12	{S-R}{S-C}{S-C}{S-C}{S-T_2}	20
{S-R}{S-T_1}{S-C}{S-T_2}	12	{S-R}{S-C}{S-C}{S-C}{C-R-C}	30
{S-R}{S-T_1}{S-C}{C-R-C}	18	{S-R}{S-C}{S-C}{S-T_2}{S-T_2}	18
{S-R}{S-T_1}{S-T_2}{S-T_2}	6	{S-R}{S-C}{S-C}{S-T_2}{C-R-C}	36
{S-R}{S-T_1}{S-T_2}{C-R-C}	12	{S-R}{S-C}{S-C}{C-R-C}{C-R-C}	36
{S-R}{S-T_1}{C-R-C}{C-R-C}	12	{S-R}{S-C}{S-T_2}{S-T_2}{S-T_2}	12
{S-R}{G_3-R}{G_3-R}	3	{S-R}{S-C}{S-T_2}{S-T_2}{C-R-C}	27
{S-R}{G_3-R}{R-R-C}	2	{S-R}{S-C}{S-T_2}{C-R-C}{C-R-C}	36
{S-R}{G_3-R}{S-S}{S-S}	6	{S-R}{S-C}{C-R-C}{C-R-C}{C-R-C}	30
{S-R}{G_3-R}{S-S}{S-C}	12	{S-R}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	5
{S-R}{G_3-R}{S-S}{S-T_2}	8	{S-R}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	12
{S-R}{G_3-R}{S-S}{C-R-C}	12	{S-R}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	18
{S-R}{G_3-R}{S-C}{S-C}	12	{S-R}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	20
{S-R}{G_3-R}{S-C}{S-T_2}	12	{S-R}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	15
{S-R}{G_3-R}{S-C}{C-R-C}	18	{S-T_1}{S-T_1}{S-T_1}	4
{S-R}{G_3-R}{S-T_2}{S-T_2}	6	{S-T_1}{S-T_1}{G_3-R}	8
{S-R}{G_3-R}{S-T_2}{C-R-C}	12	{S-T_1}{S-T_1}{R-R-C}	3
{S-R}{G_3-R}{C-R-C}{C-R-C}	12	{S-T_1}{S-T_1}{S-S}{S-S}	9
{S-R}{R-R-C}{R-R-C}	1	{S-T_1}{S-T_1}{S-S}{S-C}	18
{S-R}{R-R-C}{S-S}{S-S}	3	{S-T_1}{S-T_1}{S-S}{S-T_2}	12
{S-R}{R-R-C}{S-S}{S-C}	6	{S-T_1}{S-T_1}{S-S}{C-R-C}	18
{S-R}{R-R-C}{S-S}{S-T_2}	4	{S-T_1}{S-T_1}{S-C}{S-C}	18
{S-R}{R-R-C}{S-S}{C-R-C}	6	{S-T_1}{S-T_1}{S-C}{S-T_2}	18
{S-R}{R-R-C}{S-C}{S-C}	6	{S-T_1}{S-T_1}{S-C}{C-R-C}	27
{S-R}{R-R-C}{S-C}{S-T_2}	6	{S-T_1}{S-T_1}{S-T_2}{S-T_2}	9
{S-R}{R-R-C}{S-C}{C-R-C}	9	{S-T_1}{S-T_1}{S-T_2}{C-R-C}	18
{S-R}{R-R-C}{S-T_2}{S-T_2}	3	{S-T_1}{S-T_1}{C-R-C}{C-R-C}	18
{S-R}{R-R-C}{S-T_2}{C-R-C}	6	{S-T_1}{G_3-R}{G_3-R}	6
{S-R}{R-R-C}{C-R-C}{C-R-C}	6	{S-T_1}{G_3-R}{R-R-C}	4
{S-R}{S-S}{S-S}{S-S}{S-S}	5	{S-T_1}{G_3-R}{S-S}{S-S}	12
{S-R}{S-S}{S-S}{S-S}{S-C}	12	{S-T_1}{G_3-R}{S-S}{S-C}	24
{S-R}{S-S}{S-S}{S-S}{S-T_2}	8	{S-T_1}{G_3-R}{S-S}{S-T_2}	16
{S-R}{S-S}{S-S}{S-S}{C-R-C}	12	{S-T_1}{G_3-R}{S-S}{C-R-C}	24
{S-R}{S-S}{S-S}{S-C}{S-C}	18	{S-T_1}{G_3-R}{S-C}{S-C}	24
{S-R}{S-S}{S-S}{S-C}{S-T_2}	18	{S-T_1}{G_3-R}{S-C}{S-T_2}	24
{S-R}{S-S}{S-S}{S-C}{C-R-C}	27	{S-T_1}{G_3-R}{S-C}{C-R-C}	36
{S-R}{S-S}{S-S}{S-T_2}{S-T_2}	9	{S-T_1}{G_3-R}{S-T_2}{S-T_2}	12
{S-R}{S-S}{S-S}{S-T_2}{C-R-C}	18	{S-T_1}{G_3-R}{S-T_2}{C-R-C}	24
{S-R}{S-S}{S-S}{C-R-C}{C-R-C}	18	{S-T_1}{G_3-R}{C-R-C}{C-R-C}	24
{S-R}{S-S}{S-C}{S-C}{S-C}	20	{S-T_1}{R-R-C}{R-R-C}	2
{S-R}{S-S}{S-C}{S-C}{S-T_2}	24	{S-T_1}{R-R-C}{S-S}{S-S}	6
{S-R}{S-S}{S-C}{S-C}{C-R-C}	36	{S-T_1}{R-R-C}{S-S}{S-C}	12
{S-R}{S-S}{S-C}{S-T_2}{S-T_2}	18	{S-T_1}{R-R-C}{S-S}{S-T_2}	8
{S-R}{S-S}{S-C}{S-T_2}{C-R-C}	36	{S-T_1}{R-R-C}{S-S}{C-R-C}	12

Table Continues on Next Page

Table 4.4: Enumeration of all possible contact combinations between points, spheres, lines, cylinders, and planes resulting in a dimension of zero (continued).

Enumeration of Possible Contact Combinations (continued)			
Combination class[1]	No[2]	Combination class[1]	No[2]
$\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	12	$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	9
$\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	12	$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	18
$\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	18	$\{G_3\text{-}R\}\{G_3\text{-}R\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	18
$\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	6	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{R\text{-}R\text{-}C\}$	2
$\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	12	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}S\}\{S\text{-}S\}$	6
$\{S\text{-}T_1\}\{R\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	12	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}S\}\{S\text{-}C\}$	12
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}$	10	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}S\}\{S\text{-}T_2\}$	8
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}$	24	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	12
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}$	16	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	12
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	24	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	12
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}$	36	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	18
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}$	36	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	6
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	54	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	12
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	18	$\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	12
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	36	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}$	10
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	36	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}$	24
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	40	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}$	16
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	48	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	24
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	72	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}$	36
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	36	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}$	36
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	72	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	54
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	72	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	18
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	16	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	36
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	36	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	36
$\{S\text{-}T_1\}\{S\text{-}S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	48	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	40
$\{S\text{-}T_1\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	40	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	48
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	30	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	72
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	40	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	36
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	60	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	72
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	36	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	72
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	72	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	16
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	72	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	36
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	24	$\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	48
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	54	$\{G_3\text{-}R\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	40
$\{S\text{-}T_1\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	72	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}$	30
$\{S\text{-}T_1\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	60	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}$	40
$\{S\text{-}T_1\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	10	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	60
$\{S\text{-}T_1\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	24	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	36
$\{S\text{-}T_1\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	36	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	72
$\{S\text{-}T_1\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	40	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	72
$\{S\text{-}T_1\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	30	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	24
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{G_3\text{-}R\}$	4	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	54
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{R\text{-}R\text{-}C\}$	3	$\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	72
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}S\}$	9	$\{G_3\text{-}R\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	60
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}C\}$	18	$\{G_3\text{-}R\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}$	10
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}S\}\{S\text{-}T_2\}$	12	$\{G_3\text{-}R\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}$	24
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}S\}\{C\text{-}R\text{-}C\}$	18	$\{G_3\text{-}R\}\{S\text{-}T_2\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	36
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}C\}$	18	$\{G_3\text{-}R\}\{S\text{-}T_2\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	40
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}C\}\{S\text{-}T_2\}$	18	$\{G_3\text{-}R\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}\{C\text{-}R\text{-}C\}$	30
$\{G_3\text{-}R\}\{G_3\text{-}R\}\{S\text{-}C\}\{C\text{-}R\text{-}C\}$	27	$\{R\text{-}R\text{-}C\}\{R\text{-}R\text{-}C\}\{R\text{-}R\text{-}C\}$	1
Table Continues on Next Page			

Table 4.4: Enumeration of all possible contact combinations between points, spheres, lines, cylinders, and planes resulting in a dimension of zero (continued).

Enumeration of Possible Contact Combinations (continued)			
Combination class[1]	No[2]	Combination class[1]	No[2]
{R-R-C}{R-R-C}{S-S}{S-S}	3	{S-S}{S-S}{S-S}{S-S}{S-C}{S-T_2}	30
{R-R-C}{R-R-C}{S-S}{S-C}	6	{S-S}{S-S}{S-S}{S-S}{S-C}{C-R-C}	45
{R-R-C}{R-R-C}{S-S}{S-T_2}	4	{S-S}{S-S}{S-S}{S-S}{S-T_2}{S-T_2}	15
{R-R-C}{R-R-C}{S-S}{C-R-C}	6	{S-S}{S-S}{S-S}{S-S}{S-T_2}{C-R-C}	30
{R-R-C}{R-R-C}{S-C}{S-C}	6	{S-S}{S-S}{S-S}{S-S}{C-R-C}{C-R-C}	30
{R-R-C}{R-R-C}{S-C}{S-T_2}	6	{S-S}{S-S}{S-S}{S-C}{S-C}{S-C}	40
{R-R-C}{R-R-C}{S-C}{C-R-C}	9	{S-S}{S-S}{S-S}{S-C}{S-C}{S-T_2}	48
{R-R-C}{R-R-C}{S-T_2}{S-T_2}	3	{S-S}{S-S}{S-S}{S-C}{S-C}{C-R-C}	72
{R-R-C}{R-R-C}{S-T_2}{C-R-C}	6	{S-S}{S-S}{S-S}{S-C}{S-T_2}{S-T_2}	36
{R-R-C}{R-R-C}{C-R-C}{C-R-C}	6	{S-S}{S-S}{S-S}{S-C}{S-T_2}{C-R-C}	72
{R-R-C}{S-S}{S-S}{S-S}{S-S}	5	{S-S}{S-S}{S-S}{S-C}{C-R-C}{C-R-C}	72
{R-R-C}{S-S}{S-S}{S-S}{S-C}	12	{S-S}{S-S}{S-S}{S-T_2}{S-T_2}{S-T_2}	16
{R-R-C}{S-S}{S-S}{S-S}{S-T_2}	8	{S-S}{S-S}{S-S}{S-T_2}{S-T_2}{C-R-C}	36
{R-R-C}{S-S}{S-S}{S-S}{C-R-C}	12	{S-S}{S-S}{S-S}{S-T_2}{C-R-C}{C-R-C}	48
{R-R-C}{S-S}{S-S}{S-C}{S-C}	18	{S-S}{S-S}{S-S}{C-R-C}{C-R-C}{C-R-C}	40
{R-R-C}{S-S}{S-S}{S-C}{S-T_2}	18	{S-S}{S-S}{S-C}{S-C}{S-C}{S-C}	45
{R-R-C}{S-S}{S-S}{S-C}{C-R-C}	27	{S-S}{S-S}{S-C}{S-C}{S-C}{S-T_2}	60
{R-R-C}{S-S}{S-S}{S-T_2}{S-T_2}	9	{S-S}{S-S}{S-C}{S-C}{S-C}{C-R-C}	90
{R-R-C}{S-S}{S-S}{S-T_2}{C-R-C}	18	{S-S}{S-S}{S-C}{S-C}{S-T_2}{S-T_2}	54
{R-R-C}{S-S}{S-S}{C-R-C}{C-R-C}	18	{S-S}{S-S}{S-C}{S-C}{S-T_2}{C-R-C}	108
{R-R-C}{S-S}{S-C}{S-C}{S-C}	20	{S-S}{S-S}{S-C}{S-C}{C-R-C}{C-R-C}	108
{R-R-C}{S-S}{S-C}{S-C}{S-T_2}	24	{S-S}{S-S}{S-C}{S-T_2}{S-T_2}{S-T_2}	36
{R-R-C}{S-S}{S-C}{S-C}{C-R-C}	36	{S-S}{S-S}{S-C}{S-T_2}{S-T_2}{C-R-C}	81
{R-R-C}{S-S}{S-C}{S-T_2}{S-T_2}	18	{S-S}{S-S}{S-C}{S-T_2}{C-R-C}{C-R-C}	108
{R-R-C}{S-S}{S-C}{S-T_2}{C-R-C}	36	{S-S}{S-S}{S-C}{C-R-C}{C-R-C}{C-R-C}	90
{R-R-C}{S-S}{S-C}{C-R-C}{C-R-C}	36	{S-S}{S-S}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	15
{R-R-C}{S-S}{S-T_2}{S-T_2}{S-T_2}	8	{S-S}{S-S}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	36
{R-R-C}{S-S}{S-T_2}{S-T_2}{C-R-C}	18	{S-S}{S-S}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	54
{R-R-C}{S-S}{S-T_2}{C-R-C}{C-R-C}	24	{S-S}{S-S}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	60
{R-R-C}{S-S}{C-R-C}{C-R-C}{C-R-C}	20	{S-S}{S-S}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	45
{R-R-C}{S-C}{S-C}{S-C}{S-C}	15	{S-S}{S-C}{S-C}{S-C}{S-C}{S-C}	42
{R-R-C}{S-C}{S-C}{S-C}{S-T_2}	20	{S-S}{S-C}{S-C}{S-C}{S-C}{S-T_2}	60
{R-R-C}{S-C}{S-C}{S-C}{C-R-C}	30	{S-S}{S-C}{S-C}{S-C}{S-C}{C-R-C}	90
{R-R-C}{S-C}{S-C}{S-T_2}{S-T_2}	18	{S-S}{S-C}{S-C}{S-C}{S-T_2}{S-T_2}	60
{R-R-C}{S-C}{S-C}{S-T_2}{C-R-C}	36	{S-S}{S-C}{S-C}{S-C}{S-T_2}{C-R-C}	120
{R-R-C}{S-C}{S-C}{C-R-C}{C-R-C}	36	{S-S}{S-C}{S-C}{S-C}{C-R-C}{C-R-C}	120
{R-R-C}{S-C}{S-T_2}{S-T_2}{S-T_2}	12	{S-S}{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}	48
{R-R-C}{S-C}{S-T_2}{S-T_2}{C-R-C}	27	{S-S}{S-C}{S-C}{S-T_2}{S-T_2}{C-R-C}	108
{R-R-C}{S-C}{S-T_2}{C-R-C}{C-R-C}	36	{S-S}{S-C}{S-C}{S-T_2}{C-R-C}{C-R-C}	144
{R-R-C}{S-C}{C-R-C}{C-R-C}{C-R-C}	30	{S-S}{S-C}{S-C}{C-R-C}{C-R-C}{C-R-C}	120
{R-R-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	5	{S-S}{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	30
{R-R-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	12	{S-S}{S-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	72
{R-R-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	18	{S-S}{S-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	108
{R-R-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	20	{S-S}{S-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	120
{R-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	15	{S-S}{S-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	90
{S-S}{S-S}{S-S}{S-S}{S-S}{S-S}	7	{S-S}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	12
{S-S}{S-S}{S-S}{S-S}{S-S}{S-C}	18	{S-S}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	30
{S-S}{S-S}{S-S}{S-S}{S-S}{S-T_2}	12	{S-S}{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	48
{S-S}{S-S}{S-S}{S-S}{S-S}{C-R-C}	18	{S-S}{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	60
{S-S}{S-S}{S-S}{S-S}{S-C}{S-C}	30	{S-S}{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	60
Table Continues on Next Page			

Table 4.4: Enumeration of all possible contact combinations between points, spheres, lines, cylinders, and planes resulting in a dimension of zero (continued).

Enumeration of Possible Contact Combinations (continued)	
Combination class[1]	No.[2]
{S-S}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	42
{S-C}{S-C}{S-C}{S-C}{S-C}{S-C}	28
{S-C}{S-C}{S-C}{S-C}{S-C}{S-T_2}	42
{S-C}{S-C}{S-C}{S-C}{S-C}{C-R-C}	63
{S-C}{S-C}{S-C}{S-C}{S-T_2}{S-T_2}	45
{S-C}{S-C}{S-C}{S-C}{S-T_2}{C-R-C}	90
{S-C}{S-C}{S-C}{S-C}{C-R-C}{C-R-C}	90
{S-C}{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}	40
{S-C}{S-C}{S-C}{S-T_2}{S-T_2}{C-R-C}	90
{S-C}{S-C}{S-C}{S-T_2}{C-R-C}{C-R-C}	120
{S-C}{S-C}{S-C}{C-R-C}{C-R-C}{C-R-C}	100
{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	30
{S-C}{S-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	72
{S-C}{S-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	108
{S-C}{S-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	120
{S-C}{S-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	90
{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	18
{S-C}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	45
{S-C}{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	72
{S-C}{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	90
{S-C}{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	90
{S-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	63
{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}	7
{S-T_2}{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}	18
{S-T_2}{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}	30
{S-T_2}{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}	40
{S-T_2}{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	45
{S-T_2}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	42
{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}{C-R-C}	28
Number of combination classes = 579	
Number of combinations = 17,465	
[1] The combination class represents a set of contact combinations that have the same group representations.	
[2] This number refers to the number of elements in the set of contact combinations that have the same group representation.	

Table 4.4: Enumeration of all possible contact combinations between points, spheres, lines, cylinders, and planes resulting in a dimension of zero (continued).

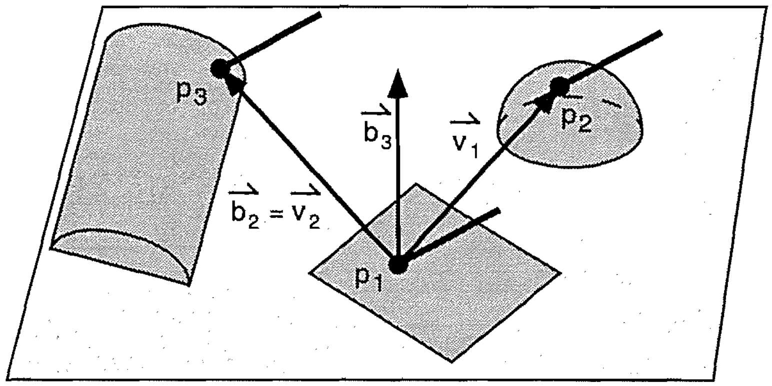


Figure 4.5: Point - fixed surface contact

4.4.1 Point-Fixed Surface Contacts

There are only three point-fixed surface contacts discussed in this dissertation. They are P/F - PL/F, P/F - S/F, and P/F - C/F. All three contacts have the group representation $\{S_o\}$. From the table, it is known that three of these contacts are necessary for the fixture to be "useful." This can be expressed mathematically as:

$$\{S_o\} \cap \{S_{o'}\} \cap \{S_{o''}\} = \{E\}. \tag{4.3}$$

The points o, o', and o'' must not be collinear, if they are then the intersection in equation 4.3 will equal $\{R_u\}$ where u is the axis through o, o', and o''.

If the surfaces are fixed, then it does not make a difference what surfaces are being touched. Hence, one P/F - PL/F contact, one P/F - S/F contact, and one P/F - P/C contact can be used as a "useful" reference fixture(see Figure 4.5).

A reference frame can be made from three non-collinear points. Given three points p_1, p_2, and p_3, we only need one point and three mutually perpendicular vectors to make a convenient reference frame. Let p_1 be the frame point, and let $\vec{v}_1 = (p_2 - p_1)$

Point - Mobile Surface Contacts	
Contact Sets	Contact Sets
6x P/F-C/M	2x P/F-C/M, 4x P/F-PL/M
5x P/F-C/M, P/F-S/M	P/F-C/M, 5x P/F-S/M
5x P/F-C/M, P/F-PL/M	P/F-C/M, 4x P/F-S/M, P/F-PL/M
4x P/F-C/M, 2x P/F-S/M	P/F-C/M, 3x P/F-S/M, 2x P/F-PL/M
4x P/F-C/M, P/F-S/M,P/F-PL/M	P/F-C/M, 2x P/F-S/M, 3x P/F-PL/M
4x P/F-C/M, 2x P/F-PL/M	P/F-C/M, P/F-S/M, 4x P/F-PL/M
3x P/F-C/M, 3x P/F-S/M	P/F-C/M, 5x P/F-PL/M
3x P/F-C/M, 2x P/F-S/M, P/F-PL/M	6x P/F-S/M
3x P/F-C/M, P/F-S/M, 2x P/F-PL/M	5x P/F-S/M, P/F-PL/M
3x P/F-C/M, 3x P/F-PL/M	4x P/F-S/M, 2x P/F-PL/M
2x P/F-C/M, 4x P/F-S/M	3x P/F-S/M, 3x P/F-PL/M
2x P/F-C/M, 3x P/F-S/M, P/F-PL/M	2x P/F-S/M, 4x P/F-PL/M
2x P/F-C/M, 2x P/F-S/M, 2x P/F-PL/M	P/F-S/M, 5x P/F-PL/M
2x P/F-C/M, P/F-S/M, 3x P/F-PL/M	6x P/F-PL/M

Table 4.5: The 28 point - mobile surface contact sets.

and $\vec{v}_2 = (p_3 - p_1)$. Then, our three mutually perpendicular vectors can be $\vec{b}_3 = \vec{v}_1 \times \vec{v}_2$, $\vec{b}_2 = \vec{v}_2$, and $\vec{b}_1 = \vec{b}_2 \times \vec{b}_3$ (see Figure 4.5).

4.4.2 Point - Mobile Surface Contacts

Point - Mobile Surface Contacts are commonly used in the literature on reference fixture design. For example, Duffie et al. [10] used point contacts to a mobile spheres and McCallion et al. [23] used point contacts to mobile planes (in the form of a cube). However, there are many more combinations that have not been discussed. In fact there are 28 combinations using just mobile plane, spheres, and cylinders. They are listed in Table 4.5.

Of the 28 sets listed in the table, not all of them are practical because they require an extra surface than needed. For example, a fixture containing two point touches to a cylinder, a sphere, and a plane is impractical because four touches to the cylinder can replace the two touches to the cylinder and the two touches to the plane. Therefore, a fixture could be made with less surfaces, making it more practical to build. If the "impractical" fixtures are eliminated then we end up with 12 "practical" fixtures. These fixtures are illustrated in Figures 4.6 and 4.7.

All of these contact combinations have one correct solution but most have a finite number of solutions and the incorrect ones must be eliminated by using one or several extra contact points. For example, the fixture containing three spheres and six P/F-S/M contacts (see Figure 4.6) can have 8 mathematical solutions. It may be necessary to touch each sphere one additional time to reduce the solution set to one answer. Figure 4.8 shows a geometric illustration of the three sphere problem.

Each one of the examples in Figures 4.6 and 4.7 require a significant amount of algebraic computation to find the solution. Therefore, each case will not be solved algebraically. However, algebraic solutions for finding a sphere, cylinder, and plane using point contacts is given in the next section.

4.5 Locating Surfaces in Space

In order to use any of the point - mobile surface fixtures, it is necessary to find geometric information using only touches to the surfaces of the fixture. In this section, methods for determining the location of a plane, sphere, and cylinder using a finite

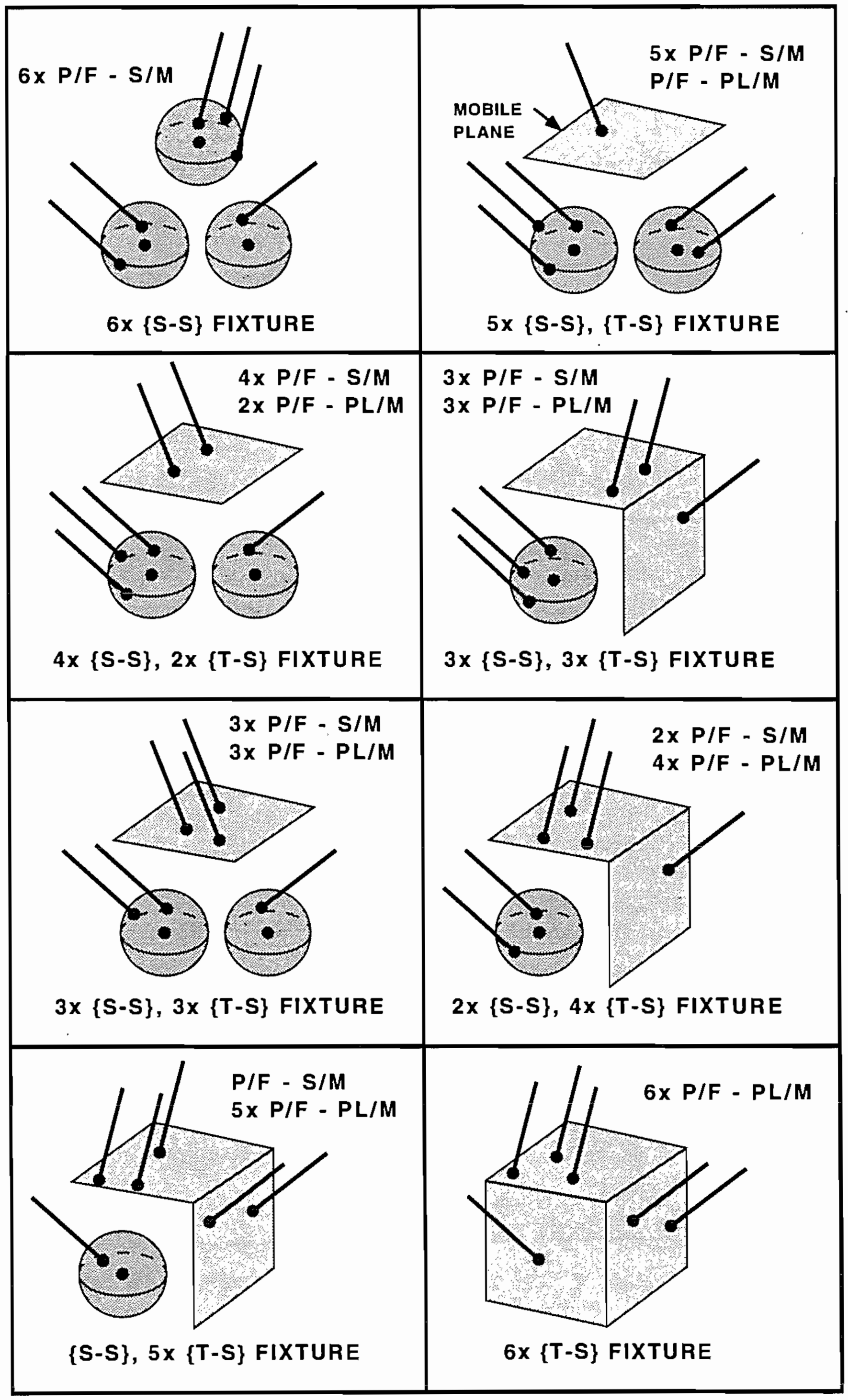


Figure 4.6: Point - surface contacts without cylinders

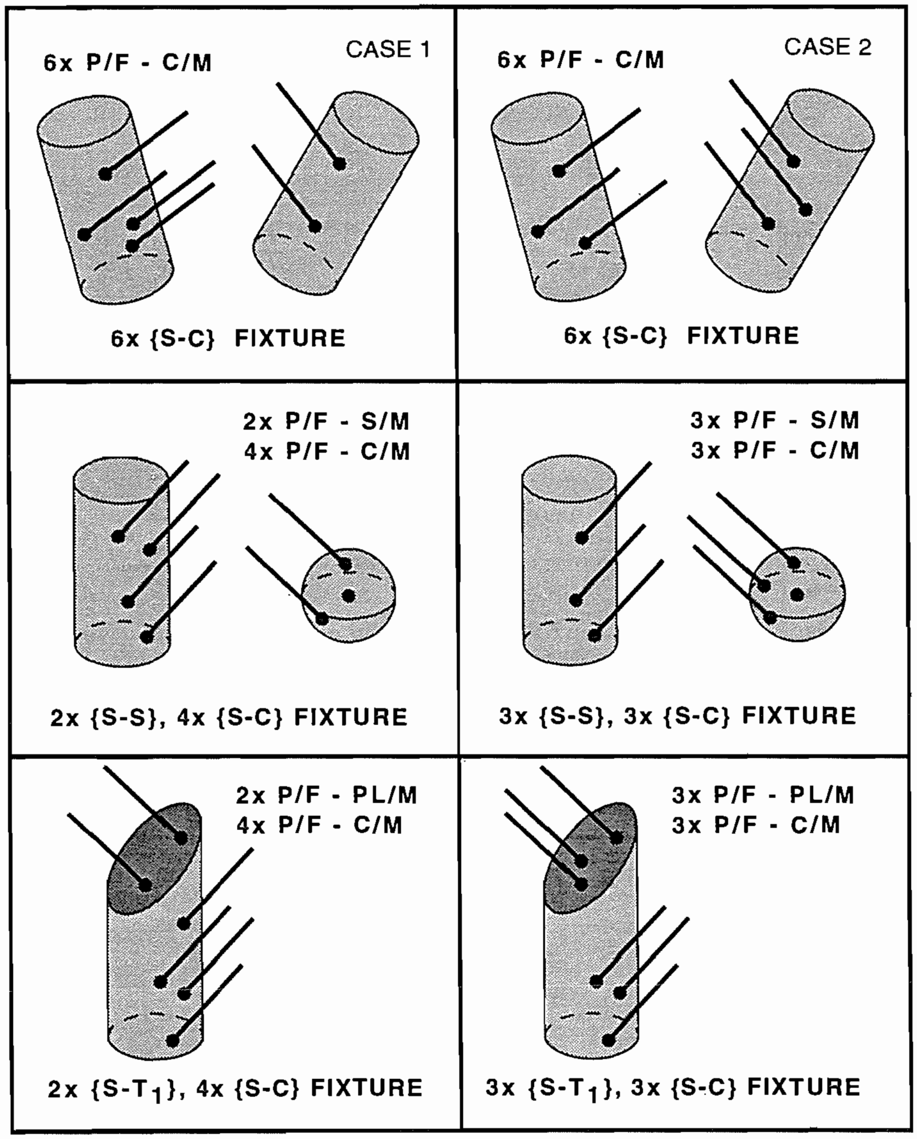


Figure 4.7: Point - surface contacts with cylinders

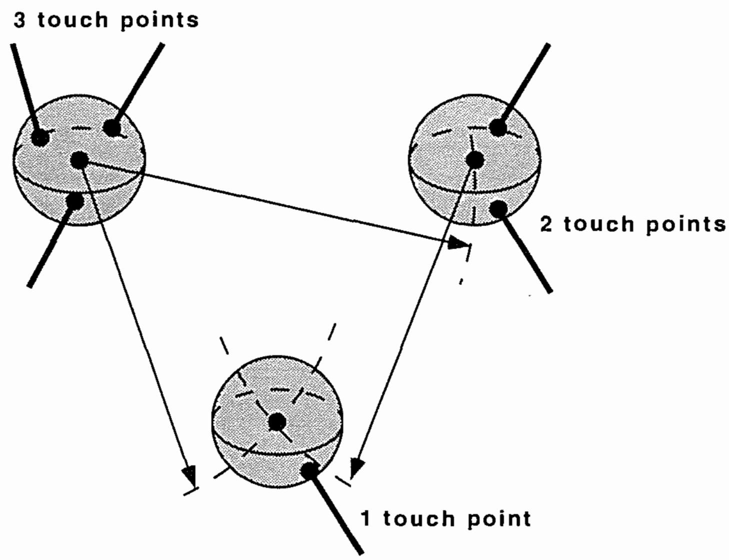


Figure 4.8: Three sphere fixture geometric solution

number of points on the surface are developed and examples of each case are given.

4.5.1 Determination of a Plane's Location Using 3 Points

The location of a plane, in general, can be found in three dimensional Euclidean space if the location of three points on the plane's surface are known (see Figure 4.9). If the points are collinear or if any of the points are coincident with each other, then a unique solution does not exist. An algebraic method is given in the next section that will uniquely determine the location of a plane given three point locations. An example is also given that uses the algebraic method found.

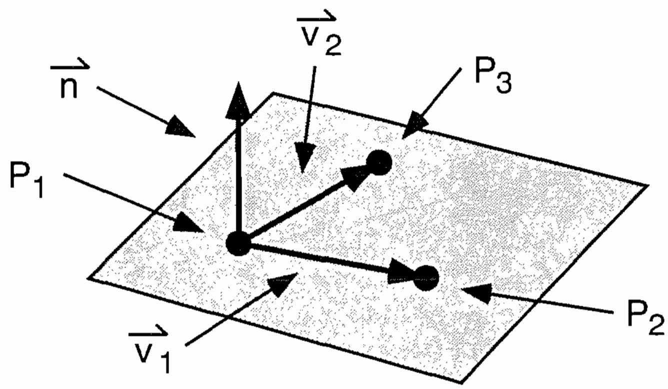


Figure 4.9: Three points on a plane

4.5.1.1 Algebraic Method

Let points p_1, p_2, and p_3 be in a plane where $p_i = (x_i, y_i, z_i)$. We now define two vectors as $\vec{v}_1 = p_2 - p_1$ and $\vec{v}_2 = p_3 - p_1$. Let $\vec{n} = \vec{v}_1 \times \vec{v}_2 = < a, b, c >$. Since $\vec{v}_1$ and $\vec{v}_2$ are parallel to the plane, $\vec{n}$ must be normal to the plane. From algebraic geometry [39], the equation of the plane is:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \tag{4.4}$$

or, in general form

$$ax + by + cz + d = 0 \tag{4.5}$$

where $d = -ax_1 - by_1 - cz_1$.

4.5.1.2 An Example

If given the following points:

$$p_1 = (1,2,3), \quad p_2 = (-1,0,-1), \quad and \quad p_3 = (3,2,1)$$

then we will find that

$$\vec{v}_1 =< -2,-2,-4 >, \quad \vec{v}_2 =< 2,0,-2 >, \quad and$$

$$\vec{n} =< -2,-2,-4 > \times < 2,0,-2 >=< 4,-12,4 > .$$

The equation for the plane is:

$$(4)(x-(1)) + (-12)(y-(2)) + (4)(z-(3)) = 0.$$

The plane in general form is:

$$4x - 12y + 4z + 8 = 0. \tag{4.6}$$

4.5.2 Determination of a Sphere's Location Using 3 Points

The location of a sphere of known radius can be found in three dimensional Euclidean space if the location of three points on the sphere's surface are known (see Figure 4.10). When three points are used, there will be two possible sphere locations that will contain all three points. If a touch probe is being used, then possibly one of those solutions can be eliminated leaving only the real solution. An algebraic method is given in the next section that will determine the location of a sphere. An example is also given that uses the algebraic method found.

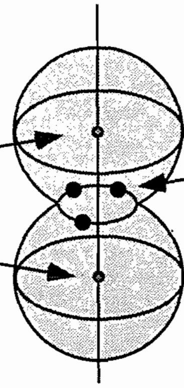


Figure 4.10: Three points on a sphere gives two possible solutions

4.5.2.1 Algebraic Method

The general equation for a sphere in space is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2 = 0, \tag{4.7}$$

where (a, b, c) is the center of the sphere and r is the radius of the sphere.

Given the points $p1$, $p2$, and $p3$ on the surface of a sphere of radius r, the following three equations must be satisfied:

$$(x_1-a)^2 + (y_1-b)^2 + (z_1-c)^2 = r^2, \tag{4.8}$$

$$(x_2-a)^2 + (y_2-b)^2 + (z_2-c)^2 = r^2, \tag{4.9}$$

$$(x_3-a)^2 + (y_3-b)^2 + (z_3-c)^2 = r^2. \tag{4.10}$$

If we subtract equation 4.9 from equation 4.8 we obtain

$$(x_2-x_1)a + (y_2-y_1)b + (z_2-z_1)c + 0.5({x_1}^2 - {x_2}^2 + {y_1}^2 - {y_2}^2 + {z_1}^2 - {z_2}^2) = 0. \tag{4.11}$$

If we subtract equation 4.10 from equation 4.8 we obtain

$$(x_3 - x_1)a + (y_3 - y_1)b + (z_3 - z_1)c + 0.5({x_1}^2 - {x_3}^2 + {y_1}^2 - {y_3}^2 + {z_1}^2 - {z_3}^2) = 0. \tag{4.12}$$

Using equation 4.11 and equation 4.12, we can solve for a and b in terms of c. If we do so, we obtain

$$\begin{aligned} b = [2(-x_2z_1 + x_3z_1 + x_1z_2 - x_3z_2 - x_1z_3 + x_2z_3)c + {x_1}^2x_2 - x_1{x_2}^2 - {x_1}^2x_3 \\ +{x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2 + x_2{y_1}^2 - x_3{y_1}^2 - x_1{y_2}^2 + x_3{y_2}^2 + x_1{y_3}^2 - x_2{y_3}^2 \\ +x_2{z_1}^2 - x_3{z_1}^2 - x_1{z_2}^2 + x_3{z_2}^2 + x_1{z_3}^2 - x_2{z_3}^2]M \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} a = [2(y_2z_1 - y_3z_1 - y_1z_2 + y_3z_2 + y_1z_3 - y_2z_3)c + {x_2}^2y_1 - {x_3}^2y_1 - {x_1}^2y_2 \\ +{x_3}^2y_2 - {y_1}^2y_2 + y_1{y_2}^2 + {x_1}^2y_3 - {x_2}^2y_3 + {y_1}^2y_3 - {y_2}^2y_3 - y_1{y_3}^2 + y_2{y_3}^2 \\ -y_2{z_1}^2 + y_3{z_1}^2 + y_1{z_2}^2 - y_3{z_2}^2 - y_1{z_3}^2 + y_2{z_3}^2]M \end{aligned} \tag{4.14}$$

where $M = 1/[2(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3)]$.

Equation 4.13 and equation 4.14 can be substituted into equation 4.8 to leave a quadratic equation in terms of the variable c. This equation can be solved to give two values for c. These values can be substituted back into equation 4.13 and equation 4.14 to obtain to center points for the sphere. Let the center points be $Center_1 = (a_1, b_1, c_1)$ and $Center_2 = (a_2, b_2, c_2)$.

If a touch sensor is being used to make contact with the sphere then the knowledge of the orientation of the sensor relative to the two possible spheres may possibly

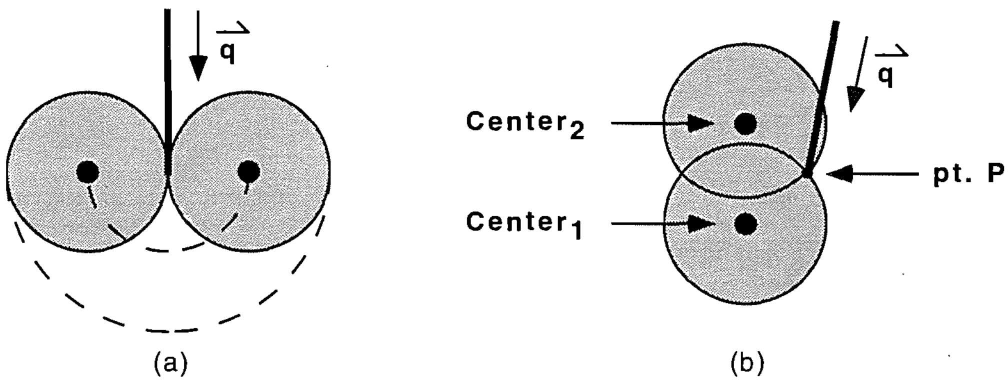


Figure 4.11: Elimination of One of Two Sphere Centers.

eliminate one of the two mathematical solutions, hence, leaving the correct solution. Let $\vec{q}$ be the vector pointing in the direction of the touch sensor. Let p be the location of the touch to the surface of the sphere. From Figure 4.11a, it can be seen that the center of the sphere must be located "lower" than point p. From Figure 4.11b, one of the two spheres cannot be possible because the touch sensor must intersect that sphere to reach point p. Therefore, we can eliminate the sphere that does not satisfy the following equation:

$$(p - Center_i) \cdot \vec{q} < 0. \tag{4.15}$$

If both sphere centers pass this test for all three points then another point will be necessary to eliminate one of the sphere centers. We now will do an example.

4.5.2.2 An Example

If we let $r = 2$, $p_1 = (1,0,0)$, $p_2 = (0,1,0)$, $p_3 = (-1,0,0)$, and $\vec{q} =< -4,1,1 >$, then equations 4.8, 4.9, and 4.10 become

$$(1-a)^2 + (b)^2 + (c)^2 = 4, \tag{4.16}$$

$$(a)^2 + (1-b)^2 + (c)^2 = 4, \tag{4.17}$$

$$(-1-a)^2 + (b)^2 + (c)^2 = 4. \tag{4.18}$$

From equations 4.11 and 4.12 we find that $a = b = 0$. Plugging these results into equation 4.16, we find $c = \pm\sqrt{3}$. Therefore our possible center points are $Center_1 = (0,0,+\sqrt{3})$ or $Center_2 = (0,0,-\sqrt{3})$. Using $\vec{q}$, $Center_1$ cannot be possible because

$$\begin{aligned} [P_1 - Center_1] \cdot \vec{q} &= [(1,0,0) - (0,0,\sqrt{3})] \cdot < -4,1,1 > \\ < 1,0,-\sqrt{3} > \cdot < 1,1,-4 > &= 1 + 4\sqrt{3} > 0. \end{aligned} \tag{4.19}$$

Hence, the sphere center must be $Center_2 = (0,0,-\sqrt{3})$.

4.5.3 Determination of a Cylinder's Location Using 5 Points

The Location of a cylinder of known radius, in general, can be found in three dimensional Euclidean space if five points on the surface of the cylinder are known. There are many different ways of determining the location of the cylinder, however, a method used by Schaal [37] develops algebraic equations using the least amount of variables. This method has been extended for application here. In the next section the algebraic equations for finding a cylinder are derived. An example using this derived method is also given.

4.5.3.1 Equation Formulation

In order to find the location of a cylinder in space using a finite number of points on its surface, a general equation for a cylinder needs to be formulated. The standard equation for a cylinder is

$$(x-a)^2 + (y-b)^2 = r^2. \tag{4.20}$$

This equation is for a right cylinder with its center line in the z axis direction and through the point (a, b). This is not a general equation for a cylinder with an arbitrary orientation. Equation 4.20 can be transformed into a general equation by application of a coordinate transformation. However, after the application of the coordinate transformation the equation is no longer easy to use. Schaal's [37] description of a cylinder is for arbitrary orientation, and it is relatively simple. Therefore, it is the basis of the method we use.

In Figure 4.12, a cylinder is given with the following properties: r is the radius of the cylinder, x and o are points on the surface of the cylinder, a and a' are points on the axis of the cylinder, $\vec{s}$ is a vector in the direction of the axis of the cylinder, α is the angle between $\vec{s}$ and $\overrightarrow{(x-a)}$, $\vec{x}$ is a vector from point o to point x, and $\vec{f}$ is a vector from point o to the axis of the cylinder where $\vec{f} \perp \vec{s}$. From Figure 4.12 and

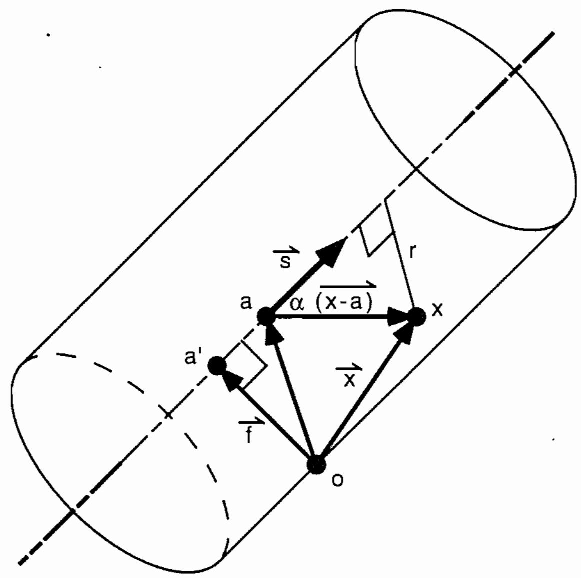


Figure 4.12: Analytical model of a cylinder

vector analysis it is obvious that

$$r = |\overrightarrow{(x-a)}| \sin\alpha. \tag{4.21}$$

Also from vector analysis we know that the cross product between two vectors is equal in magnitude to the multiplication of the magnitudes of each vector times the sine of the angle between the two vectors. Using this property and equation 4.21 we get the following,

$$|\overrightarrow{(x-a)} \times \vec{s}| = |\overrightarrow{(x-a)}||\vec{s}| \sin\alpha = (|\overrightarrow{(x-a)}| \sin\alpha)|\vec{s}| = r|\vec{s}|. \tag{4.22}$$

Letting $[\vec{v}]^2 = \vec{v} \cdot \vec{v}$, equation 4.22 can be written as,

$$[\overrightarrow{(x-a)} \times \vec{s}]^2 = r^2[\vec{s}]^2 \quad or \quad [\overrightarrow{(x-a)} \times \vec{s}]^2 - r^2[\vec{s}]^2 = 0. \tag{4.23}$$

Equation 4.23 is a general equation for a cylinder. We will now proceed to change the equation to only leave the variable $\vec{s}$.

The left side of equation 4.23, substituting $\overrightarrow{(x-a)} = \vec{x} - \vec{f}$, can be written as

$$[\overrightarrow{(x-a)} \times \vec{s}]^2 = [(\vec{x} - \vec{f}) \times \vec{s}] \cdot [(\vec{x} - \vec{f}) \times \vec{s}]. \tag{4.24}$$

Using the vector equation

$$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}), \tag{4.25}$$

$[(\vec{x} - \vec{f}) \times \vec{s}]$ can be written as

$$[(\vec{x} - \vec{f}) \times \vec{s}] = [(\vec{x} \times \vec{s}) - (\vec{f} \times \vec{s})]. \tag{4.26}$$

Substituting $\vec{\delta} = \vec{f} \times \vec{s}$ into the right side of equation 4.26, we obtain

$$[(\vec{x} \times \vec{s}) - (\vec{f} \times \vec{s})] = [(\vec{x} \times \vec{f}) - \vec{\delta}]. \tag{4.27}$$

Substituting equation 4.27 into equation 4.24, we obtain

$$[\overrightarrow{(x-a)} \times \vec{s}]^2 = [(\vec{x} \times \vec{f}) - \vec{\delta}] \cdot [(\vec{x} \times \vec{f}) - \vec{\delta}]. \tag{4.28}$$

Using the vector equation

$$(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}, \tag{4.29}$$

equation 4.28 can be written as

$$[\overrightarrow{(x-a)} \times \vec{s}]^2 = [\vec{x} \times \vec{s}]^2 + [\vec{\delta}]^2 - 2[(\vec{x} \times \vec{s}) \cdot \vec{\delta}]. \tag{4.30}$$

We already know that

$$[(\vec{x} \times \vec{s}) \cdot \vec{\delta}] = [(\vec{x} \times \vec{s}) \cdot (\vec{f} \times \vec{s})] \tag{4.31}$$

from our definition of δ. Using the vector equation

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}), \tag{4.32}$$

equation 4.31 can be written as

$$[(\vec{x} \times \vec{s}) \cdot (\vec{f} \times \vec{s})] = (\vec{x} \cdot \vec{f})(\vec{s} \cdot \vec{s}) - (\vec{x} \cdot \vec{s})(\vec{s} \cdot \vec{f}). \tag{4.33}$$

If we let $\vec{s} \perp \vec{f}$ then $(\vec{s} \cdot \vec{f}) = 0$. This changes equation 4.33 to

$$[(\vec{x} \times \vec{s}) \cdot (\vec{f} \times \vec{s})] = (\vec{x} \cdot \vec{f})(\vec{s} \cdot \vec{s}) = (\vec{x} \cdot \vec{f})[\vec{s}]^2. \tag{4.34}$$

Substituting equation 4.34 into equation 4.30, we obtain

$$[\overrightarrow{(x - a)} \times \vec{s}]^2 = [\vec{x} \times \vec{s}]^2 + [\vec{\delta}]^2 - 2(\vec{x} \cdot \vec{f})[\vec{s}]^2. \tag{4.35}$$

Substituting equation 4.35 into equation 4.23, we obtain

$$[\vec{x} \times \vec{s}]^2 + [\vec{\delta}]^2 - 2(\vec{x} \cdot \vec{f})[\vec{s}]^2 - r^2[\vec{s}]^2 = 0. \tag{4.36}$$

Since we let $\vec{f} \perp \vec{s}$, then the magnitude of $\vec{f}$ must be equal to the radius of the cylinder. Therefore, $|\vec{\delta}| = |(\vec{f} \times \vec{s})| = |\vec{f}||\vec{s}|$. Hence,

$$[\vec{\delta}]^2 - r^2[\vec{s}]^2 = 0. \tag{4.37}$$

Using equation 4.37, equation 4.36 becomes

$$[\vec{x} \times \vec{s}]^2 - 2(\vec{x} \cdot \vec{f})[\vec{s}]^2 = 0. \tag{4.38}$$

Let x_1, x_2, x_3, x_4, and x_5 be the points on the surface of the cylinder. Let x_5 be the point o, and let $p_i = x_i - x_5$ for $i = 1$ *to* 4. With these changes, equation 4.38 can be written as

$$(\vec{p_i} \cdot \vec{f}) - \frac{1}{2[\vec{s}]^2}[\vec{p_i} \times \vec{s}]^2 = 0 \quad for\ i = 1\ \ to\ \ 4. \tag{4.39}$$

Expanding equation 4.39 into a matrix formula using p_1, p_2, and p_3, we obtain

$$\begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} \\ p_{2_x} & p_{2_y} & p_{2_z} \\ p_{3_x} & p_{3_y} & p_{3_z} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} - \frac{1}{2[\vec{s}]^2} \begin{bmatrix} (\vec{p}_1 . \times \vec{s})^2 \\ (\vec{p}_2 \times \vec{s})^2 \\ (\vec{p}_3 \times \vec{s})^2 \end{bmatrix} = 0. \tag{4.40}$$

If we let

$$M = \begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} \\ p_{2_x} & p_{2_y} & p_{2_z} \\ p_{3_x} & p_{3_y} & p_{3_z} \end{bmatrix}, \tag{4.41}$$

then

$$M^{-1} = \frac{1}{det[M]} \begin{bmatrix} (\vec{p}_2 \times \vec{p}_3)_x & (\vec{p}_3 \times \vec{p}_1)_x & (\vec{p}_1 \times \vec{p}_2)_x \\ (\vec{p}_2 \times \vec{p}_3)_y & (\vec{p}_3 \times \vec{p}_1)_y & (\vec{p}_1 \times \vec{p}_2)_y \\ (\vec{p}_2 \times \vec{p}_3)_z & (\vec{p}_3 \times \vec{p}_1)_z & (\vec{p}_1 \times \vec{p}_2)_z \end{bmatrix}. \tag{4.42}$$

Multiplying both sides of equation 4.40 by M^{-1} gives

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \frac{1}{2[\vec{s}]^2\ det[M]} \begin{bmatrix} (\vec{p}_2 \times \vec{p}_3)_x & (\vec{p}_3 \times \vec{p}_1)_x & (\vec{p}_1 \times \vec{p}_2)_x \\ (\vec{p}_2 \times \vec{p}_3)_y & (\vec{p}_3 \times \vec{p}_1)_y & (\vec{p}_1 \times \vec{p}_2)_y \\ (\vec{p}_2 \times \vec{p}_3)_z & (\vec{p}_3 \times \vec{p}_1)_z & (\vec{p}_1 \times \vec{p}_2)_z \end{bmatrix} \begin{bmatrix} (\vec{p}_1 \times \vec{s})^2 \\ (\vec{p}_2 \times \vec{s})^2 \\ (\vec{p}_3 \times \vec{s})^2 \end{bmatrix}. \tag{4.43}$$

Let $\vec{n}_1 = \vec{p}_2 \times \vec{p}_3$, $\vec{n}_2 = \vec{p}_3 \times \vec{p}_1$, and $\vec{n}_3 = \vec{p}_1 \times \vec{p}_2$. Substituting $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ into equation 4.43 and multiplying both sides of the equation by $\vec{s}$ we get

$$[\vec{s}]\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \frac{[\vec{s}]}{2[\vec{s}]^2\ det[M]} \begin{bmatrix} n_{1_x} & n_{2_x} & n_{3_x} \\ n_{1_y} & n_{2_y} & n_{3_y} \\ n_{1_z} & n_{2_z} & n_{3_z} \end{bmatrix} \begin{bmatrix} (\vec{p}_1 \times \vec{s})^2 \\ (\vec{p}_2 \times \vec{s})^2 \\ (\vec{p}_3 \times \vec{s})^2 \end{bmatrix}. \tag{4.44}$$

We know that $(\vec{s} \cdot \vec{f}) = 0$, therefore the both sides of equation 4.44 must be equal to zero. The right side of the equation now becomes

$$\begin{bmatrix} s_x & s_y & s_z \end{bmatrix} \begin{bmatrix} n_{1_x} & n_{2_x} & n_{3_x} \\ n_{1_y} & n_{2_y} & n_{3_y} \\ n_{1_z} & n_{2_z} & n_{3_z} \end{bmatrix} \begin{bmatrix} (\vec{p}_1 \times \vec{s})^2 \\ (\vec{p}_2 \times \vec{s})^2 \\ (\vec{p}_3 \times \vec{s})^2 \end{bmatrix} = 0. \tag{4.45}$$

Equation 4.45 describes a cylinder in three dimensional Euclidean space using only four points on the surface and the vector $\vec{s}$. The vector $\vec{s}$ describes the direction of the center line of the cylinder. It can be denoted as $< s_x, s_y, s_z >$. The actual magnitude of the vector is not important for our case, therefore we can set one of the vector components equal to one. Let $s_z = 1$. For this case, vector $\vec{s}$ must not be parallel to the $x - y$ plane, if it is then s_x or s_y will go to infinity during a calculation of $\vec{s}$. If this happens then either s_x or s_y should be set to one instead of s_z. Note, it is unlikely that a vector will have any directional components equal to zero using an actual robot end-effector frame, hence, any real calculations should work with $s_z = 1$.

Using $\vec{s} =< s_x, s_y, 1 >$ means that there are two unknowns s_x and s_y, yet we only have one equation, equation 4.45. Therefore, another equation is necessary for the calculation of $\vec{s}$. If we add another point on the surface, $\vec{p}_4$, to equation 4.40, then we end up with

$$\begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} \\ p_{2_x} & p_{2_y} & p_{2_z} \\ p_{3_x} & p_{3_y} & p_{3_z} \\ p_{4_x} & p_{4_y} & p_{4_z} \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} - \frac{1}{2[\vec{s}]^2} \begin{bmatrix} (\vec{p}_1 \times \vec{s})^2 \\ (\vec{p}_2 \times \vec{s})^2 \\ (\vec{p}_3 \times \vec{s})^2 \\ (\vec{p}_4 \times \vec{s})^2 \end{bmatrix} = 0. \qquad (4.46)$$

From linear algebra [35], Equation 4.46 is only valid if

$$\det \begin{bmatrix} p_{1_x} & p_{1_y} & p_{1_z} & (\vec{p}_1 \times \vec{s})^2 \\ p_{2_x} & p_{2_y} & p_{2_z} & (\vec{p}_2 \times \vec{s})^2 \\ p_{3_x} & p_{3_y} & p_{3_z} & (\vec{p}_3 \times \vec{s})^2 \\ p_{4_x} & p_{4_y} & p_{4_z} & (\vec{p}_4 \times \vec{s})^2 \end{bmatrix} = 0. \qquad (4.47)$$

Equation 4.47 is a second order equation. Using equation 4.45 and equation 4.47 the vector $\vec{s}$ is, in general, solvable using $s_z = 1$. The two equations can be written in the general form

$$a{s_x}^3 + b{s_y}^3 + c{s_x}^2 s_y + d s_x {s_y}^2 + e{s_x}^2 + f{s_y}^2 + g s_x s_y + h s_x + i s_y + j = 0 \qquad (4.48)$$

and

$$k{s_x}^2 + l{s_y}^2 + m s_x s_y + n s_x + p s_y + q = 0. \qquad (4.49)$$

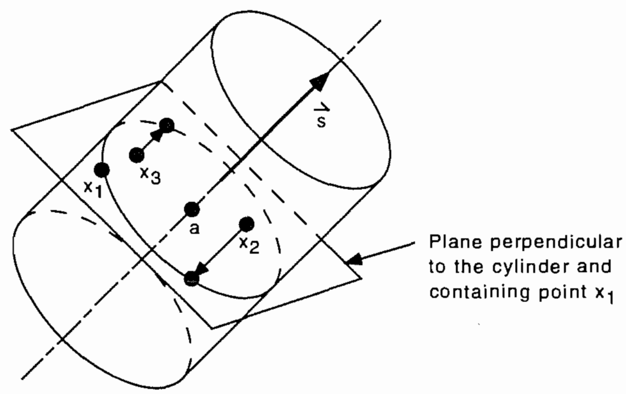


Figure 4.13: Finding a point on the center line of a cylinder

Equations 4.48 and 4.49 can be combined to form one equation of degree six in only one variable using Sylvester's Method [6]. This equation can be solved numerically to obtain six possible solutions for one of the variables, s_x or s_y. Using equation 4.49 the other variable can be found. Therefore, we will have six possible solutions for the cylinder axis direction $\vec{s}$. We still need to find a point on the center line of the cylinder to define the location of the cylinder in space. We will do this by using three of the points on the surface to create a circle on a plane perpendicular to the cylinder and find the center of this circle, as shown in Figure 4.13.

When we find the center of the circle, we can also calculate the radius of the circle which is also the radius of the cylinder. Since we know the radius of the cylinder already, we can eliminate the cases where the calculated radius does not match the actual radius. This should leave only the correct cylinder result. In the next section, an example is given to illustrate the cylinder finding process.

List of Possible Cylinder Locations		
$\vec{s}$	Radius of cyl.	Point on center line
$< -1, -1, 1 >$	$r = 1.08012$	$(0.8333, 0.8333, 0.6667)$
$< 0, 0, 1 >$	$r = 1.00000$	$(0, 0, 0)$
$< 1, 1, 1 >$	$r = 1.08012$	$(0.1667, 0.1667, 0.6667)$
$< 0.2928, 0.7071, 1 >$	$r = 1.10587$	$(-0.038, -0.092, 0.3694)$
$< 1.7071, -0.7071, 1 >$	$r = 0.95912$	$(0.4668, -0.193, -0.773)$

Table 4.6: List of possible cylinder locations.

4.5.3.2 An Example

Let $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x_3 = (-1,0,1)$, $x_4 = (0,-1,1)$, and $x_5 = (1,0,2)$ be five points on the surface of a cylinder of radius $r = 1$. This corresponds to a cylinder where $\vec{s} =< 0,0,1 >$. Since the answer is known, the results of the example can be verified. Using equation $\vec{p_i} = x_i - x_5$ for i = 1 to 4, we find $\vec{p_1} =< 0,0,-2 >$, $\vec{p_2} =< -1,1,-2 >$, $\vec{p_3} =< -2,0,-1 >$, and $\vec{p_4} =< -1,-1,-1 >$.

Using the values for $\vec{p_i}$, equation 4.45 written in the form of equation 4.48 is

$$8s_x - 2{s_x}^3 + 8s_xs_y - 6{s_x}^2s_y - 8{s_y}^2 - 2s_x{s_y}^2 + 2{s_y}^3. \tag{4.50}$$

Using the values for $\vec{p_i}$, equation 4.47 written in the form of equation 4.49 is

$$8s_x - 8{s_x}^2 - 8s_y + 8{s_y}^2. \tag{4.51}$$

Using a mathematics equation solver, equations 4.50 and 4.51 are solved for $\vec{s}$. Five solutions exist for this case, and they are: $\vec{s} =< -1,-1,1 >$, $\vec{s} =< 0,0,1 >$, $\vec{s} =< 1,1,1 >$, $\vec{s} =< 0.292893, 0.707107, 1 >$, and $\vec{s} =< 1.70711, -0.707107, 1 >$.

Using the values of $\vec{s}$ we calculated the values for the radius of the cylinder and a point on the cylinder's center line. Table 4.6 shows the results. The value of the radius for the five cases varies from 0.95912 to 1.10587. We know the radius for this particular case is exactly one. Hence, the case that has a radius that is very close to one should be the correct cylinder. The case with $\vec{s} =< 0,0,0 >$ has a mathematically determined radius of 1.000000000000. Therefore, it is the correct cylinder. This matches the expected result.

If the actual radius of the cylinder was not known ahead of time, then an additional point on the cylinder would be needed to eliminate the incorrect solutions. However, this would not happen when working with reference fixtures because the user designs the fixture.

Chapter 5

Practical Fixtures

In Chapter four, fixture geometries consisting of point surface contacts were analyzed in detail because of their practicality and simplicity. Both mobile-surface and fixed-surface contacts were studied. We found that three point fixed-surface contacts are enough to uniquely define a coordinate frame on the fixture. We also found twelve fixture geometries consisting of point mobile-surface contacts that have potential as reference fixtures.

In this chapter, two fixture geometries are studied further for application in referencing environments. Both fixture geometries are developed to the point were a prototype fixture can be constructed. One of the fixture geometries uses three point fixed-surface contacts, and the other uses seven point mobile-surface contacts.

The three point fixed-surface contact fixture design consists of a planar fixed-surface and a tripod shaped touch probe. To facilitate the design of this fixture, several different types of position sensing surfaces are discussed. Moreover, the design of a robust touch probe is considered. Finally, a detailed example of the use of this fixture design is given with an error analysis.

The seven point mobile-surface contact fixture consists of a planer surface and a cylindrical surface. A mathematical approach for constructing a coordinate frame on the fixture using the geometries of the cylinder and plane is given. This approach uses the five point method for finding the location of a cylinder in space that was described in Chapter four. A numerical example is given to further illustrate the use of this approach. Finally, the actual design of the fixture is considered, two possible mechanical designs are discussed, and a touch sensing electrical circuit is described.

5.1 A Cylinder-Plane Fixture

Twelve fixture designs involving point-mobile surface contacts were discussed in Chapter four. Almost all of these fixtures had two or more separate components. For example, the three sphere fixture required three separated spheres to work. On the other hand, The three planes in the three-plane fixture did not have to be separated. The three planes could be combined to form a cube. McCallion and Pham [23] described this cubical fixture in their paper on reference fixture design. Only one other of the twelve fixture designs can easily be made into one piece, the cylinder-plane fixture design (see Figure 5.1). Because of its simplicity, it will be analyzed further for practical application as a reference fixture.

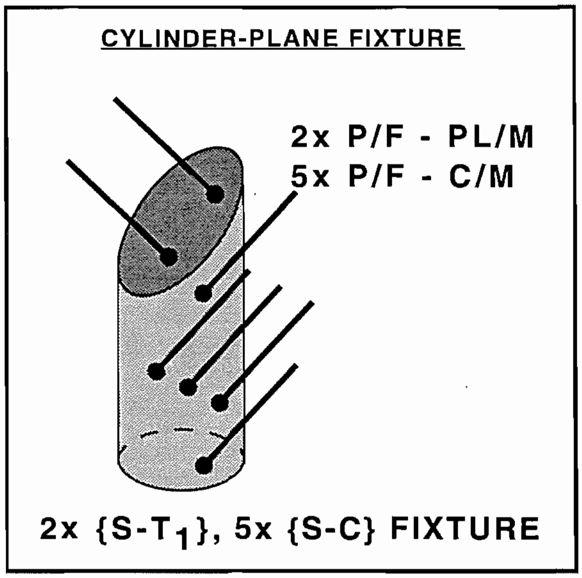


Figure 5.1: Cylinder-plane fixture

The cylinder-plane fixture can be used to *uniquely* determine the location of the fixture's frame using point-mobile surface contacts. From Chapter four, it is known that five points will uniquely find the location of a cylinder of known radius. Also, from Chapter four, we should be able to find the plane, using the cylinder as a guide, using two points on the plane's surface. Therefore, we should be able to find the fixture's frame in seven points. The technique for finding the fixture's frame is discussed in the next section.

5.1.1 Mathematical Determination of the Frame Location

Given five points on the surface of a cylinder of known radius, the location of the cylinder can be found using the technique described in Chapter four. The result will be a vector along the center line of the cylinder, and a point on the center line of

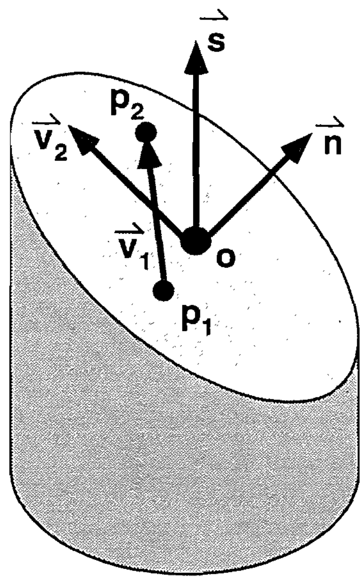


Figure 5.2: Analysis of the coordinate frame attached to the plane-cylinder fixture

the cylinder. Let $\vec{s} =< s_x, s_y, s_z >$ and $q = (q_x, q_y, q_z)$ be this normalized vector and point, respectively.

With the center line of the cylinder known, the location of the plane can be found if two points on its surface are known. Let those two points be p_1 and p_2. Let $\vec{v}_1 = p_2 - p_1 =< x_{21}, y_{21}, z_{21} >$ and $\vec{v}_2 =< x, y, 1 >$. $\vec{v}_2$ is the vector that points up the incline of the plane (see Figure 5.2).

If the incline of the plane is set to a specific angle (other than zero and ninety degrees) relative to the cylinder's center line, then, using vector algebra [18], the components of $\vec{v}_2$ can be found. For simplicity we will let the angle between the cylinder's center line and the plane be 45 degrees. Assuming that $\vec{v}_2$ is already known, the normal to the plane can be determined by taking the cross product between $\vec{v}_1$ and $\vec{v}_2$. Let $\vec{n} = \vec{v}_1 \times \vec{v}_2$. With the the angle between the plane and center line equal to 45 degrees, the angle between $\vec{n}$ and $\vec{s}$ and the angle between $\vec{v}_2$ and $\vec{s}$ must both be 45 degrees.

From vector algebra [18], the following equation is known:

$$\frac{\vec{k} \cdot \vec{l}}{|\vec{k}||\vec{l}|} = \cos(\theta) \tag{5.1}$$

where θ is the angle between $\vec{k}$ and $\vec{l}$. Using equation 5.1, $\vec{s}$, $\vec{n}$, and the angle between them, the following equation is obtained,

$$\frac{\vec{n} \cdot \vec{s}}{|\vec{n}|} = \cos(45) = \frac{1}{\sqrt{2}}. \tag{5.2}$$

Taking the square of equation 5.2, we obtain

$$2(\vec{n} \cdot \vec{s})^2 = \vec{n} \cdot \vec{n}. \tag{5.3}$$

The same procedure can be applied to $\vec{v}_2$ and $\vec{s}$ to get

$$2(\vec{v}_2 \cdot \vec{s})^2 = \vec{v}_2 \cdot \vec{v}_2. \tag{5.4}$$

Expanding equation 5.3 and equation 5.4 using the components of $\vec{n}$,$\vec{s}$, and $\vec{v}_2$ we get

$$2(-s_y x_{21} + s_y z_{21} x - s_z y_{21} x + s_z x_{21} y + s_x y_{21} - s_x z_{21} y)^2 =$$
$$(z_{21} x - x_{21})^2 + (x_{21} y - y_{21} x)^2 + (y_{21} - z_{21} y)^2 \tag{5.5}$$

and

$$2(s_z + s_x x + s_y y)^2 = 1 + x^2 + y^2. \tag{5.6}$$

Equations 5.5 and 5.6 can be solved to find $\vec{v}_2$. With $\vec{v}_2$ known, the normal to the plane can be calculated, and, therefore, the equation of the plane can be found using the normal and one of the points on the plane.

With the location of the plane and cylinder known, it is now possible to define a frame on the fixture. $\vec{v}_2$, $\vec{n}$, and $\vec{v}_2 \times \vec{n}$ are three mutually perpendicular vectors. To define a frame on the fixture we also need a point. The point where the center line of the cylinder intersects the plane will work. We denote this point as o. The center line of the cylinder can be written as

$$(o_x, o_y, o_z) = (q_x, q_y, q_z) + l < s_x, s_y, s_z > \tag{5.7}$$

where $l \in \mathbb{R}$. If equation 5.7 is substituted into the equation for the plane, then a value for l can be found that, when substituted back into equation 5.7, will determine the location of the point of intersection. In the next section, an example is given to demonstrate these calculations.

5.1.2 An Example

Given a cylinder-plane fixture, let $\vec{s} =< 0, 0, 1 >$ and $q = (0, 0, -5)$ where $\vec{s}$ and q define the center line of the cylinder (these values could be found using five points on the surface of the cylinder and the methods described in Chapter four). Let $p_1 = (0, -1, -1)$ and $p_2 = (1, 1, 1)$ be two points on the surface of the plane. Since

$\vec{v}_1 = p_2 - p_1$, $\vec{v}_1 =< 1, 2, 2 >$. Using these values, equation 5.5 becomes

$$(y - 2x)^2 = (2 - 2y)^2 + (2x - 1)^2 \tag{5.8}$$

and equation 5.6 becomes

$$x^2 + y^2 - 1 = 0. \tag{5.9}$$

Solving equations 5.8 and 5.9 with a numerical analysis package, we find $\vec{v}_2 =< 0, 1, 1 >$. Using $\vec{v}_1$ and $\vec{v}_2$, the normal to the plane is $< 0, -1, 1 >$, and the equation for the plane is $y - z = 0$. The point where the center line intersects the plane can now be determined. The equation for the center line of the cylinder using q and $\vec{s}$ is

$$(o_x, o_y, o_z) = (0, 0, -5) + l < 0, 0, 1 > \tag{5.10}$$

Substituting equation 5.10 into the equation of the plane, we get $(0+l0)-(-5+l) = 0$. Therefore, $l = 5$, and the point of intersection (o_x, o_y, o_z) is $(0, 0, 0)$. It is now possible to define the frame on the fixture using $\vec{v}_2$, $\vec{n}$, $\vec{v}_2 \times \vec{n}$, and the intersection point $(0, 0, 0)$.

5.1.3 Design of a Cylinder-Plane Fixture

We now have the mathematical means to find a coordinate frame using the geometric elements that compose the cylinder-plane fixture. However, this does not describe how to design and build a working fixture. In this section, the design of a fixture in terms of its electrical and mechanical components is described.

A point - mobile surface fixture must be touched several times for its location to be determined. In the case of the cylinder-plane fixture, five touches to the cylinder and two touches to the plane are needed. In general, the robot will be holding a touch sensing device and it will make contact with the fixture. At this instance, the location of the contact will be stored in the robot's computer for later use. This procedure will continue until all seven points are recorded.

In order to use the mathematical procedure discussed in the previous section, it is necessary to know which of the touch contacts are to the cylindrical surface and which ones are to the planar surface. The idea behind building a reference fixture is that the robot can determine the location of the fixture and, using this information, know the location of everything that the fixture is attached to. Therefore, the robot is not going to know if it is touching the cylindrical surface or the planar surface. Actually it will not know if it is even touching the fixture at all. Therefore, this information needs to be relayed to the computer during a touch contact. This could be accomplished by a person controlling the robot. This person could inform the robot's controller of what it made contact with during every contact. This, however, is a tedious task. Hence, it should be avoided.

If the fixture is designed with a little bit of intelligence, then it can indicate that a contact has been made to the cylindrical surface or planar surface. One easy way to do this is to do a conductivity test. If each surface is conductive, each surface is isolated from conductive materials, and the tip of the robot's touch probe is conductive, then during a contact to the surface, the conductivity between the touch probe and each surface can be checked to see if contact has been made. This idea is illustrated in

Figure 5.4.

For this design to work each surface must be conductive and isolated from other conductive material. Two different possibilities for doing this are shown in Figure 5.3. It is also important to eliminate any edges that could be touched that would give incorrect results. For example, the conducting planar surface in Figure 5.3b has its edges covered with an insulator so that the touch probe cannot touch at those points. If the edge was not covered, then the probe could make a conducting contact at a point that is not on the planar surface (an edge point), and, hence, sense a "bad" point.

5.2 A Three Point Fixed-Surface Contact Fixture

In Chapter four, it was found that three point-fixed surface contacts is enough to determine the location of the fixture. In this section, the design of fixtures that use these contacts are explored. Finally, we use this design knowledge in the design of a simple, practical touch sensing reference fixture.

5.2.1 Mathematical Procedure

As stated in Chapter four, a reference frame can be made from three non-collinear points. We only need one point and three mutually perpendicular vectors to make a convenient reference frame. Given three points p_1, p_2, and p_3, we can construct three

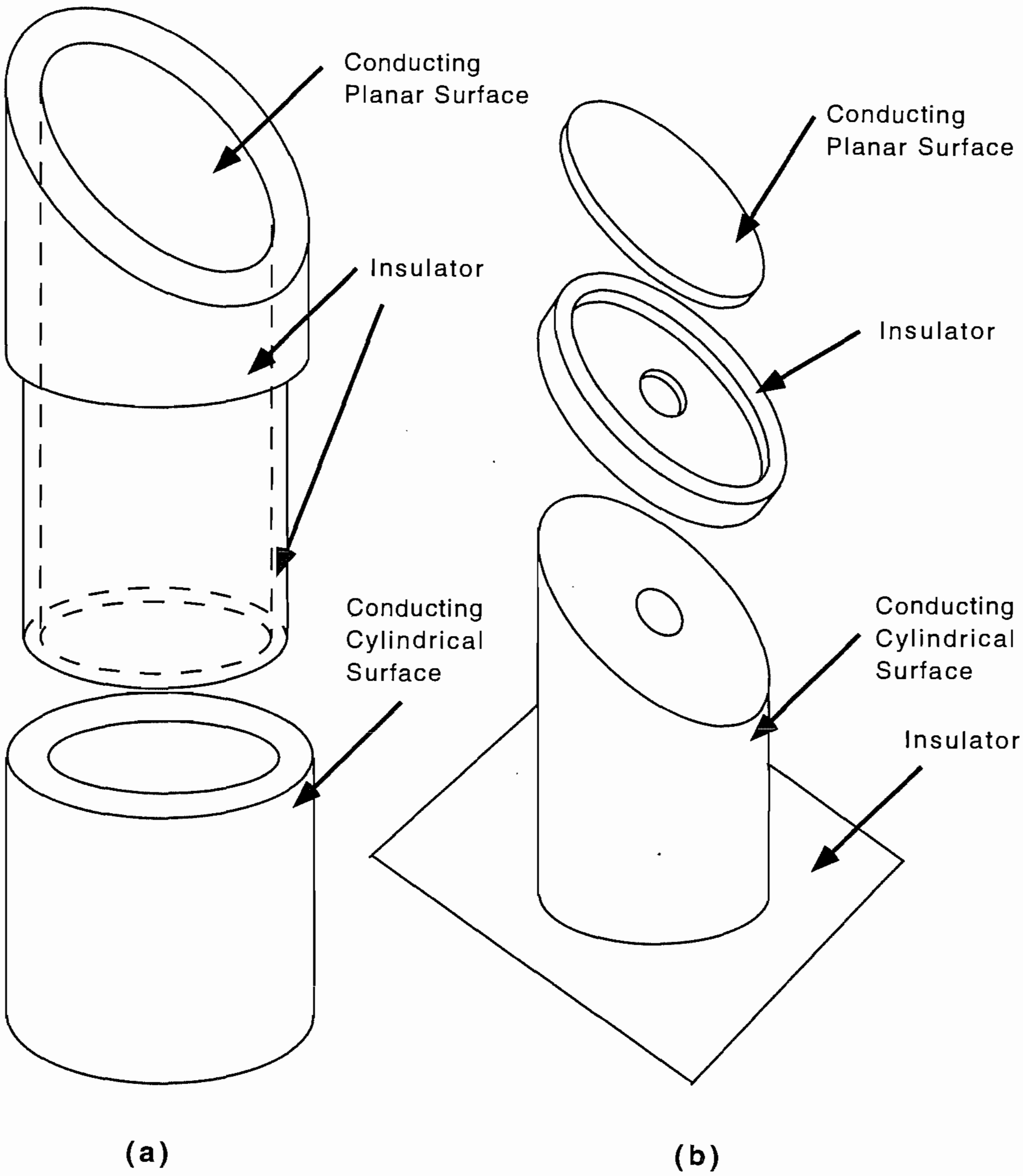


Figure 5.3: Cylinder-plane fixture designs

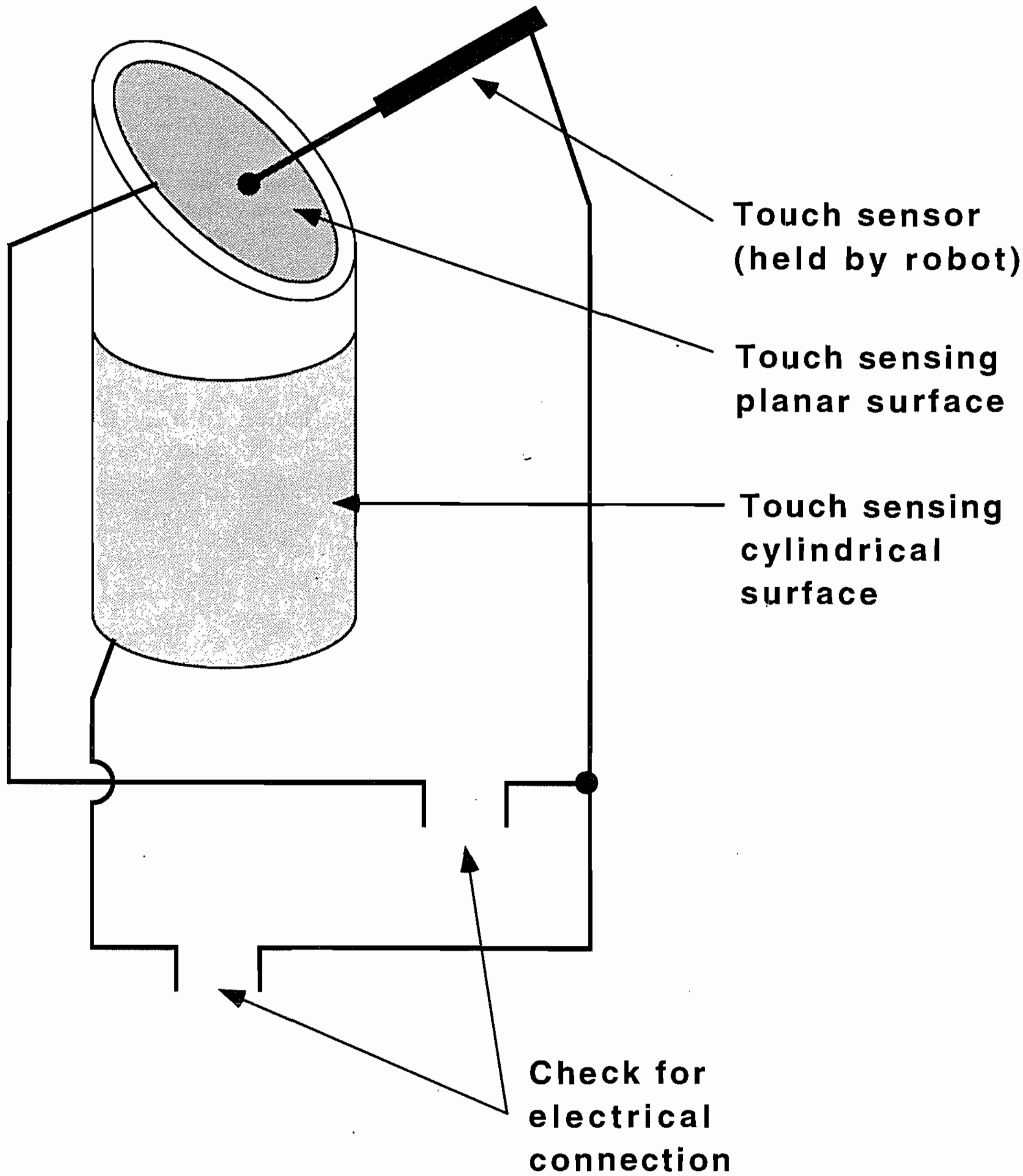


Figure 5.4: The touch sensing circuit for the cylinder-plane fixture

mutually perpendicular vectors. Let p_1 be the frame point, and let $\vec{v}_1 = (p_2 - p_1)$ and $\vec{v}_2 = (p_3 - p_1)$. Then our three mutually perpendicular vectors can be $\vec{b}_3 = \vec{v}_1 \times \vec{v}_2$, $\vec{b}_2 = \vec{v}_2$, and $\vec{b}_1 = \vec{b}_2 \times \vec{b}_3$.

5.2.2 Design of Point Fixed-Surface Contact Fixtures

The mathematics for creating a frame of reference are relatively simple. However, no methods for actually building a fixed-surface fixture have been described. In this section, several different methods for locating a touch to a surface in the surface's reference frame are described. Moreover, the design of a touch probe is also discussed.

5.2.2.1 Sensing of Contact Locations

There are several different ways of determining the location of a touch to a surface. Bicchi, Salisbury, and Brock [5] used a force-moment sensor in the base of an object to determine the location of a touch to the surface of that object. Moreover, touch sensitive computer screens are currently being used to give the location of a touch to the surface of a screen (Ormond [33]).

Force/Moment Sensor

Force/moment sensors are commonly used in robotic wrists to relay information back to the robot controller about the size of the external forces and moments being applied to the wrist of the robot. If both the force vector and moment vector are given, from the force/moment sensor, and the location of the force/moment sensor is

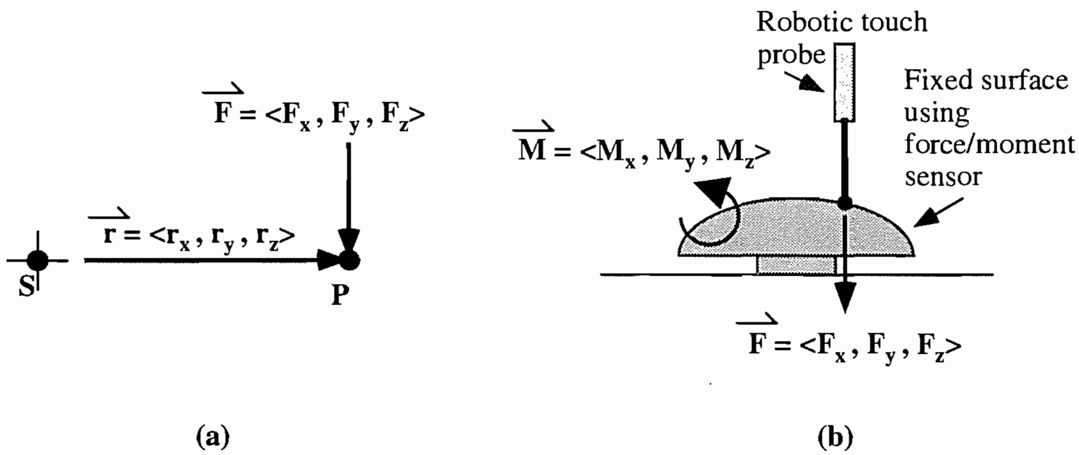


Figure 5.5: Calculation of the line of action using a force/moment sensor

known, then the direction, line of action, and magnitude of the force being applied to the wrist can be found relative to the robot.

If this same procedure is applied to a surface equipped with a force/moment sensor, then the location of a touch to this surface can be determined. The location of the touch is determined by finding the intersection between the surface and line of action. Therefore, this method can be applied to any surface of known shape as long as the line of action only intersects the surface once.

The procedure for determing the line of action is relatively simple (see Figure 5.5). Given a force vector $\vec{F}$ and a moment vector $\vec{M}$ relative to the location of the force/moment sensor, point S, we need to find $\vec{r}$ where $\vec{r}$ is the shortest position vector from point S to the line of action. The line of action, using a point-vector representation, is defined by the point, $S + \vec{r}$, and the vector, $\vec{F}$. To find $\vec{r}$, we use the following relationship between $\vec{M}$ and $\vec{F}$

$$\vec{M} = \vec{r} \times \vec{F} \tag{5.11}$$

where $\vec{r}$, for this equation, is any vector from point S to the line of action. For $\vec{r}$ to be the shortest one, it must be perpendicular to the line of action. Hence, $\vec{r}$, $\vec{F}$, and $\vec{M}$ must be mutually perpendicular. This result implies that the direction of $\vec{r}$ is the same as the direction of $\vec{F} \times \vec{M}$. It also implies that

$$|\vec{M}| = |\vec{r}||\vec{F}| \quad or \quad |\vec{r}| = \frac{|\vec{M}|}{|\vec{F}|}. \tag{5.12}$$

Using equation 5.12 and the fact that $\vec{F} \times \vec{M}$ is in the same direction as $\vec{r}$, we get

$$\frac{\vec{r}}{|\vec{r}|} = \frac{\vec{F} \times \vec{M}}{|\vec{M}||\vec{F}|}. \tag{5.13}$$

With some algebraic manipulation and a substitution for $|\vec{r}|$ using equation 5.12, equation 5.13 becomes

$$\vec{r} = \frac{\vec{F} \times \vec{M}}{|\vec{F}|^2}. \tag{5.14}$$

With equation 5.14 and point S found, the line of action is known.

Touch Sensing Screens

Several different methods for finding the location of a touch to a surface are described in Ormond [33]. Ormond describes several different technologies for sensing the location of a touch on the screen of a computer where the screen has a special cover for sensing the location. Some of the sensing methods are, discrete resistance

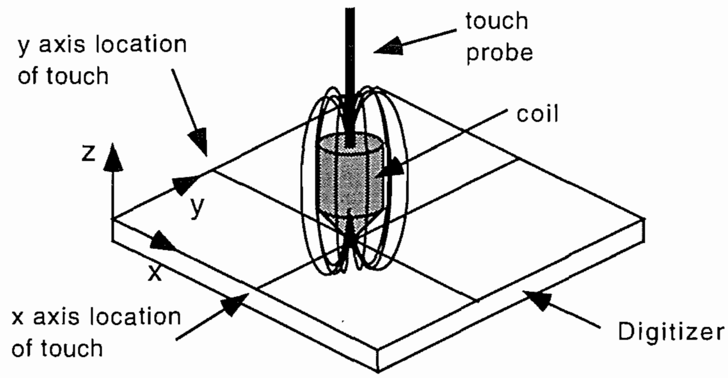


Figure 5.6: A digitizer sensing the magnetic field caused by a coil

pads, capacitance measurement pads, and scanning IR pads. Descriptions of these methods can be found in [33].

Position Measurement Using a Digitizer

Although touch sensitive screens and force-moment sensors do have potential for reference fixture design, they do have problems. The force/moment method cannot handle more than one touch at a time. If more than one touch exists, then the resulting Force and Moment measurements will be the combination of both touches, making the calculated location incorrect. The touch sensitive screens may be able to handle more than one touch, but they are not accurate enough for robotic referencing. Digitizers, however, do not have either one of these problems.

Digitizers sense an energized coil's magnetic field to determine the location of the digitizer's pointer (see Figure 5.6). The pointer contains a coil at its tip, and the coil obtains its power from the digitizer via a control/power cord. Digitizers are usually used with computers as a tool for drawing or drafting. However, they can be modified for application as a touch sensing surface. If a touch probe is equipped with a coil of the same size as contained in the pointer of the digitizer, then the digitizer would not know the difference, and it would report back the location of the probe. Moreover, if a several touch probes, each equipped with coils, touched the digitizer together and each coil was activated in a sequence, then the location of each probe would be found.

Most digitizers communicate via a RS232 port. The digitizer sends the location of the "touch" in X,Y coordinates where the range of each axis varies from 0 to 2^n where n is an integer that is digitizer model dependent. Using the size of the digitizer, the actual location in inches or millimeters can be found [40]. The accuracy of a digitizer varies from model to model. Models exists with accuracies of 0.005 inches and sizes up to 44 inches x 60 inches [32].

5.2.2.2 Touch Sensing Probe

Up until now, only the design of touch sensing surfaces have been considered, however, the touch probe is equally important. In general, the probe will be held by the robot wrist, and it will make contact with a mobile or fixed surface. In either case, the probe should be robust enough to take a small collision with the surface. One possible way to make the probe more robust in a collision is to make the tip of the probe compliant during a collision. The problem with doing this is that the probe will no longer have a very precise length. This problem can be overcome by using a spring loaded linear displacement transducer at the end of the probe. Several different linear displacement transducers exist, for example, LVDTs and variable resistors. Another possible transducer is a digital indicator.

Digital indicators are commonly used in manufacturing environments for checking the accuracy of machined parts. They can be purchased with various accuracies and strokes. They usually come equipped with an output cable that can be connected to a computer to record the measurement results. A digital indicator is ideally suited for use as a probe because it is accurate, it can be connected to a computer, it is readily available, it is designed for a tough environment, and it is relatively cheap.

5.2.3 An Actual Design

In order to create a reference fixture that uses three point fixed-surface contacts to define a frame location, we need to find a practical position sensing surface and a practical touch probe. Ideally, we want to make all three contacts at the same time to increase speed and lower complexity during use. From the earlier discussion, the digitizer and digital indicator seemed suited for this task.

Using the idea of a three coil/digitizer combination, we have designed a touch sensing tripod/digitizer fixture (see Figure 5.7). This fixture incorporates a three finger touch sensor where each finger is composed of a digital indicator with a coil at its tip. Figure 5.8 shows the tip assembly used to replace the existing tip of the digital indicator. When this sensor, or tripod, comes in contact with the digitizer, each digital indicator moves in until all of three digital indicator tips come into contact

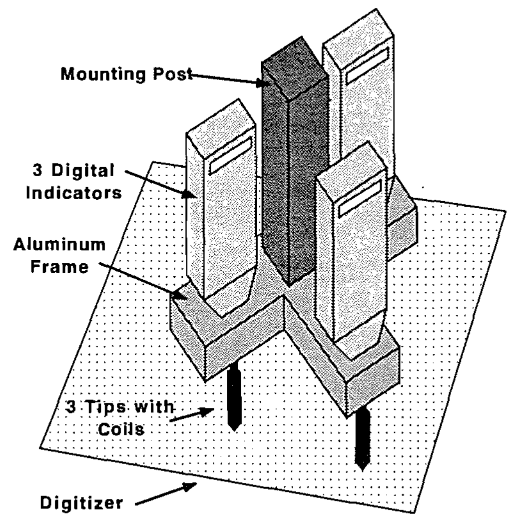


Figure 5.7: Illustration of the tripod/digitizer fixture

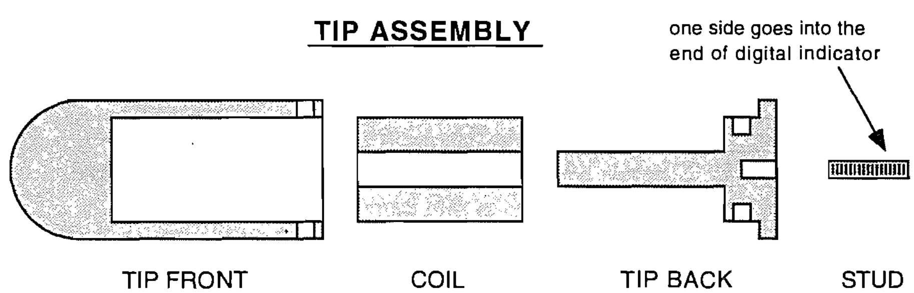


Figure 5.8: Tripod/digitizer fixture tip assembly

with the digitizer. Once in contact, each coil is energized and the location of each indicator tip is found in the frame of the digitizer. The location of the three tips is also known in the frame of the touch sensing tripod because the displacement of each digital indicator is known. Therefore, the location of the three points is known in both frames and the relative location of the tripod to the digitizer can be found.

This touch sensing tripod/digitizer fixture is built in our laboratory and is presently being tested. The digitizer and digital indicator outputs are being controlled by a single-board digital controller. A schematic of the electrical connections is shown in Figure 5.9. Figure 5.10 shows the unit being test on a milling machine. This fixture, in addition to its simplicity, has the advantage of being able to make a complete reference measurement with one touching motion and being able to make a complete reference measurement, via the digitizer, at different angles and positions.

5.2.3.1 An Example

An example is given to further illustrate the use of the tripod/digitizer reference fixture. Figure 5.11a gives the coordinate systems that we will use for the analysis of this example. The X, Y, and Z coordinate axes are connected to the tripod aluminum frame. They form a coordinate frame that we will refer to as frame F. The X'', Y'', and Z'' axes are connected to the corner of the digitizer. They form a coordinate frame that we will refer to as frame F''. The X', Y', and Z' axes are the intermediate axes created by the three contact points. They form a coordinate frame that we will

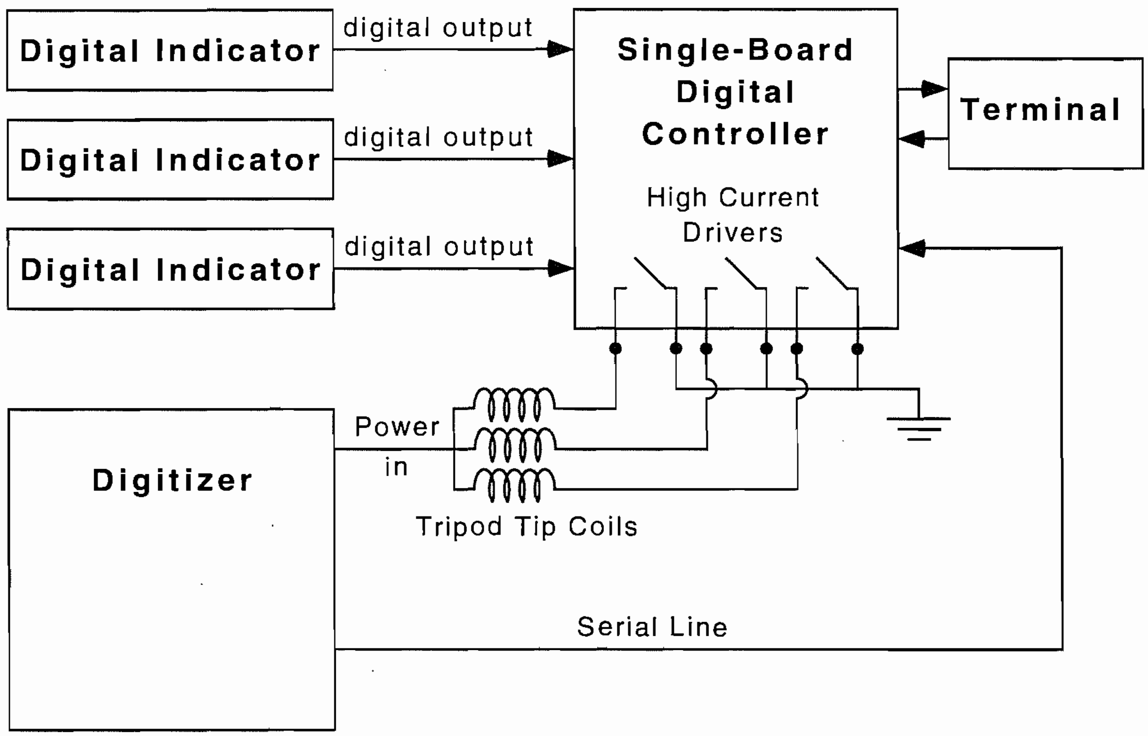


Figure 5.9: Control schematic for tripod fixture

Figure 5.10: Test setup for tripod fixture

refer to as frame F'.

The object of the analysis is to find the transformation matrix from frame F to frame F'' using the measurements made by the digital indicators, the digitizer, and the hardware dimensions. The two critical hardware dimensions are shown in Figure 5.11. Figure 5.11b shows the distance between the bottom of the aluminum frame (frame F) and the tip of each digital indicator (frame F'). Figure 5.11c shows the spacing between the three digital indicators.

For this example, let digital indicator number 1 read 0.500, digital indicator number 2 read 0.700, and digital indicator number 3 read 0.700. Also let points A, B, and C have the following coordinates in frame F'': point A = (3,4,0), point B = (4.513,2.677,0), and point C = (4.338,5.5,0). Using the given values (all in inches) and the hardware dimensions, it is possible to find the transformation matrix from frame F to frame F'' using the intermediate frame F'. The first step is to find the transformation matrix from F to F'.

The transformation matrix from F to F' can be found if one point and three vectors are known in both frames. The points A, B, and C can be determined in all the frames. In frame F the point coordinates are, point $A = [0, 0, (-3 + 0.5)]$, point $B = [2, 0, (-3 + 0.7)]$, and point $C = [0, 2, (-3 + 0.7)]$. These same points have the following coordinates in frame F', point $A = (0, 0, 0)$, point $B = (2.01, 0, 0)$, and point $C = (0, 2.01, 0)$. Using these points, three mutually perpendicular vectors can be found. Let $\vec{v}_1$ = point B - point A and $\vec{v}_2$ = point C - point A. Let these

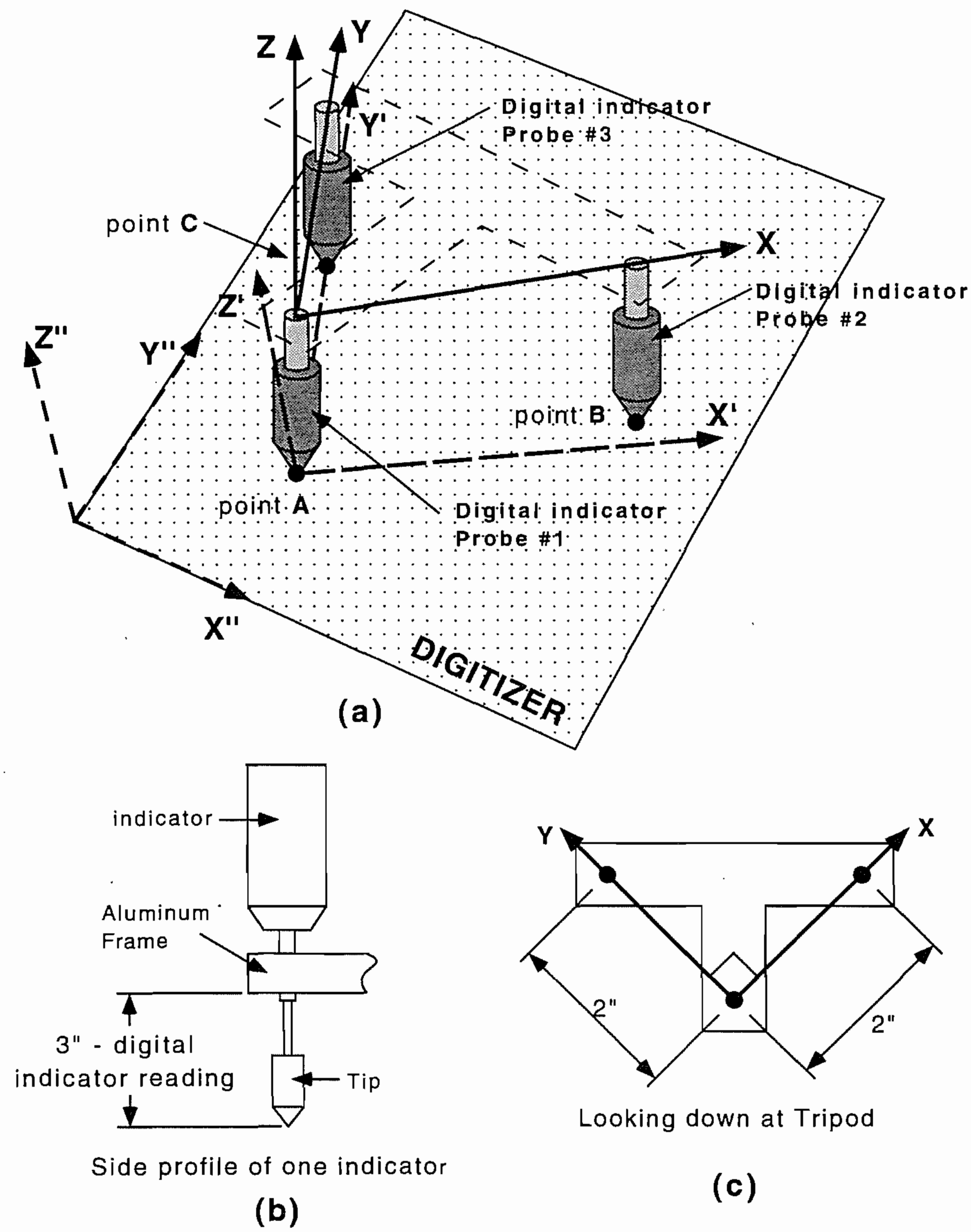


Figure 5.11: Fixture analysis

three mutually perpendicular vectors be $\vec{b}_3 = (\vec{v}_1 \times \vec{v}_2)/|\vec{v}_1 \times \vec{v}_2|$, $\vec{b}_2 = \vec{v}_2/|\vec{v}_2|$, and $\vec{b}_1 = \vec{b}_2 \times \vec{b}_3$. In frame F, $\vec{v}_1 = (2,0,-2.3) - (0,0,-2.5) =< 2,0,0.2 >$, $\vec{v}_2 = (0,2,-2.3) - (0,0,-2.5) =< 0,2,0.2 >$, $\vec{b}_1 =< 0.995086, -0.009852, 0.098523 >$, $\vec{b}_2 =< 0.000000, 0.995037, 0.995038 >$, and $\vec{b}_3 =< -0.099015, -0.099015, 0.990148 >$. In frame F' these vectors are, $\vec{v}_1 =< 2.010, 0, 0 >$, $\vec{v}_2 =< 0, 2.010, 0 >$, $\vec{b}_1 =< 1, 0, 0 >$, $\vec{b}_2 =< 0, 1, 0 >$, and $\vec{b}_3 =< 0, 0, 1 >$.

Using the three mutually perpendicular vectors and point A, the transformation matrix can be determined. Using homogeneous coordinates, a vector $< x, y, x >$ in frame F can be transformed into a vector $< x', y', z' >$ in frame F' using the following equation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & d_1 \\ a_{21} & a_{22} & a_{23} & d_2 \\ a_{31} & a_{32} & a_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}. \tag{5.15}$$

Moreover, a point (x,y,z) in frame F can be transformed into a point (x', y', z') in frame F' using the following equation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & d_1 \\ a_{21} & a_{22} & a_{23} & d_2 \\ a_{31} & a_{32} & a_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}. \tag{5.16}$$

Using equation 5.15 and the vectors $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ in frames F and F', the coefficients in the transformation matrix that represent the rotational part of the transformation (the coefficients that beginning with the letter a) can be found. Using equation 5.16 and point A in both frames, the coefficients that represent the translational part of the transformation (the coefficients that beginning with the letter d) can be found. The interested reader is referred to Hearn [14] for more information on coordinate transformations.

Solving for the coefficients, the transformation matrix from frame F to frame F', denoted as T_1, is

$$T_1 = \begin{bmatrix} 0.995086 & -0.009852 & 0.098523 & 0.246308 \\ 0.000000 & 0.995037 & 0.099504 & 0.248759 \\ -0.099015 & -0.099015 & 0.990148 & 2.475369 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}. \quad (5.17)$$

Using the transformation matrix T_1, it is now possible to transfer any point or vector frame F to frame F'. Now the transformation matrix between frame F' and frame F'' needs to be found in the same way. The location of the points A, B, and C is already known in both frames, and the necessary vectors are easy to calculate. After calculating these values, the transformation matrix from frame F' to frame F'', denoted as T_2, is

$$T_2 = \begin{bmatrix} 0.746255 & 0.665660 & 0.000000 & 3.000000 \\ -0.665660 & 0.746255 & 0.000000 & 4.000000 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}. \tag{5.18}$$

With T_1 and T_2 known, the transformation matrix from frame F to F'' can be calculated by multiplication of the two matrices. Hence, the transformation matrix, denoted as T, is

$$T = [T_2][T_1] = \begin{bmatrix} 0.742588 & 0.655004 & 0.139759 & 3.349398 \\ -0.662389 & 0.749110 & 0.008672 & 4.021680 \\ -0.099015 & -0.099015 & 0.990148 & 2.475369 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}. \tag{5.19}$$

The transformation matrix from frame F'' to frame F is simplify the inverse of matrix T.

5.2.3.2 Error Analysis

Since the tripod/digitizer fixture will be used for referencing and calibration, it is critical for the components used in the design to be as accurate as possible. The accuracy of the tripod/digitizer fixture is limited by the accuracy of the digitizer and the digital indicators used. The design developed here and shown in Figure 5.10 uses relatively inexpensive components. The digital indicators used have a stroke of one

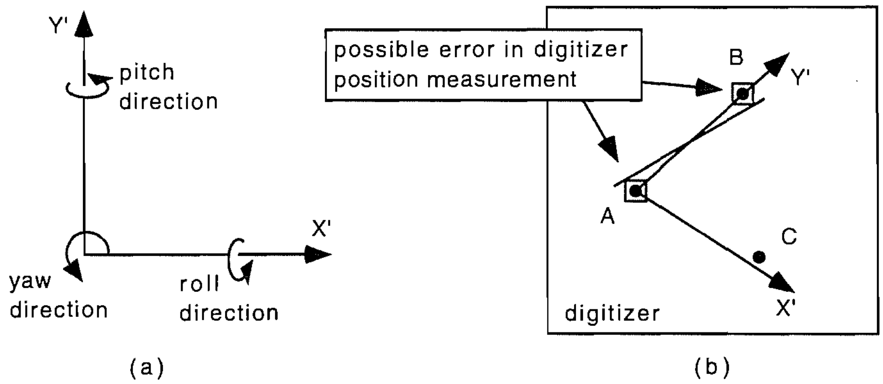


Figure 5.12: Error analysis

inch (25.4 mm) with an accuracy of 0.001 inches (0.0254 mm) over that range [25]. The digitizer used for the prototype fixture is relatively old and was found to be the limiting part for the accuracy of the prototype system developed.

We performed tests that indicated the system has an accuracy of 0.030 inches (0.762 mm) over the 11.7 inch by 11.7 inch surface (300 mm x 300 mm). It should be pointed out, however, that the accuracy of the system can be easily improved by using a more accurate digitizer. Digitizers are available with accuracies of plus or minus 0.005 inches (0.127 mm) and sizes up to 44 inches by 60 inches (1100 mm x 1500 mm) [32].

The position and orientation error in the system can be written in terms of the error in the position of the origin of frame F' and the error in the yaw, pitch, and roll angles associated with frame F'. The relationships between the angles and frame F' are shown in Figure 5.12a.

Let the maximum error in the digitizer be E_D for both the x axis and the y

axis of the digitizer. Let E_I be the maximum error in the digital indicators. If we assume that $E_D >> E_I$ and that the angle between the $X - Y$ plane and the $X' - Y'$ plane does not exceed 30 degrees, which we found to be true for our system, then the maximum position and orientation error comes when the $X - Y$ plane and the $X' - Y'$ plane are parallel.

From the information given, we already know that the position and position error for the origin of frame F' is $(0, 0, 0) \pm (E_D, E_D, E_I)$. The orientation error is a little more difficult. The method used to create frame F' relied on the position of points A and C to find the location of the Y' axis. Therefore, the digitizer error in these two points determines the error in the orientation of the Y' axis in the $X' - Y'$ plane. This is the yaw error which is shown in Figure 5.12b. Using E_D, the maximum yaw error equals $2\sqrt{2}E_D/l$ where l is the distance between digital indicator number one and number three. Using the same method, the maximum error in the roll and pitch angles are $2E_I/l$, where l is the distance between either digital indicator number one and number three or digital indicator number one and number two. For our system they are the same.

If the tripod/digitizer fixture incorporated the more accurate digitizer that was described earlier ($E_D = 0.005$), then the yaw error and the error in the position of the organ would be significantly reduced. As an example, let $E_D = 0.005$, $E_I = 0.001$, and $l = 3$ (all values are in inches). Using these values, the position error of the origin is $\pm(0.005, 0.001, 0.001)$, the yaw error is 0.004714 radians, and the roll and pitch errors are 0.0006667 radians. It should be noted that increasing the distance between the digital indicators reduces the error in the yaw, roll, and pitch angles. Hence, a more accurate design would use a very accurate digitizer in conjunction with a tripod sensor that incorporates large distances between each of the digital indicators.

Chapter 6

Conclusion

6.1 An Overview

In the introduction of this dissertation, the existing work on the design of tactile fixtures for referencing was described as limited. This dissertation addressed this issue by creating a theoretical framework for the design of tactile fixtures. In doing so, several new and interesting results were found.

In Chapter three, a new method for analysis of fixtures based on the geometric surfaces that formed them was given. This method relied upon the Euclidean group and its subgroups. To aid in the use of the Euclidean group, the complete set of continuous subgroups of the Euclidean group were found using a Lie algebra/Lie group approach. Using the Euclidean group, several propositions were introduced and proven. These propositions formed the basis of a new theory that can aid in the design of touch sensing fixtures by analysis of the continuous and finite groups that represent them. Using these propositions, fixtures involving spheres, planes, cylinders, and combinations of these geometric elements were analyzed for their "usefulness."

In Chapter four, the analysis of fixtures was taken one step further by looking at contacts needed to make a "useful" fixture. Contacts between spheres, planes, cylinders, points, and lines were studied, and group representations were found for every possible contact that could exist between these geometric elements. Using these group representations, different types of contacts were found to have equivalent group representations. This made it possible to treat different contacts as if they were the same. Using these new contact classes, all possible combinations of contacts were studied. During this enumeration, 579 contact combination classes and 17,465 actual contact combinations were found.

After completing this enumeration, a set of contact combinations, the point surface contact combinations, were examined more closely because they formed a more practical set of touch sensing fixture designs. Two types of point surface contacts were studied, combinations involving point mobile surface contacts and combinations involving fixed surface contacts. It was found that three point fixed surface contacts were enough to determine the location of a fixture in space. As for the mobile surface contacts, 28 point mobile surface combinations were found. Of these 28, 12 were explored further for referencing applications. In order to study these fixtures, analysis methods were developed for finding the location of a sphere, a cylinder, and a plane in space using only a finite number of points on each surface. It was shown that a plane needs three points to define its location, a sphere requires three or four points depending on the circumstance, and a cylinder of known radius requires five points. Of these three case, the method for determination of a cylinder's location in space using points on its surface is new. Examples were given for each of these surfaces.

Finally, in Chapter 5, using the information obtained from the analysis of the point surface contacts, two simple yet novel touch sensing fixtures for referencing were developed. One of those fixtures used a plane-cylinder geometry to uniquely locate a frame. The other fixture used a tripod shaped probe and a planar fixed surface (in the final design a digitizer was used) to uniquely locate a reference frame. In developing the later fixture, several different technologies for use as fixed surfaces were explored. The final design of the tripod/digitizer fixture is currently patent pending.

6.2 Future Research

Although, many goals were accomplished in this dissertation, several areas can be developed much further, and several of the developed ideas can be applied to other fields. Here is a list of possible areas for future research:

1. The analysis of the fixtures in this dissertation were limited to a set of specific geometric elements; if this set is expanded to include new geometric elements,

then many new and unique fixtures can be developed.

2. The mathematical approach for the enumeration of the combination classes was not very formal; in the future, it would be ideal to improve the mathematical foundation for the enumeration of these contacts.

3. The contact analysis performed in this dissertation was for the design of reference fixtures. However, many other areas, such as general fixture design and robot assembly, deal with contacts between geometric elements. Therefore, the application of this analysis to these fields would be of value for future work.

4. Finally, the continued development of simple, practical fixtures is the main intention of this research.

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